

# W Algebras from AdS/CFT Correspondence

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We consider a system of  $D5/D1$  branes in the supergravity background  $AdS_3 \times S^3 \times X$ , where  $X$  is  $T^4$  or  $K3$ . By investigating the structure of the missing states in the conformal description, we are able to extend the  $AdS/CFT$  correspondence to W algebras. As a test of this new formulation the results are compared to Hilbert schemes and more general supergravity backgrounds as deformations by D3-branes or six-dimensional Calabi-Yau manifolds.

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## 1. Introduction

The introduction of branes in addition to the classical fundamental string brought two new aspects into the low energy formulation of the different string sectors. One is the anomaly term of D-branes and its understanding in the context of K-theory. This way, M theory provides the RR-charges with a natural intersection form and thus a lattice structure in K-theory [1]. A seemingly different ansatz pursues matrix theory. For each string sector there exists a matrix or “little string” theory, which is build up from an infinite set of branes in the infinite momentum frame of M theory. Following the *AdS/CFT* conjecture [2], these six-dimensional low energy limits are related to  $AdS_3 \times S^3 \times X$  supergravity backgrounds [3,4], where  $X$  is either  $T^4$  or  $K3$ . In this article we will mainly address the second point and investigate the supergravity theory of  $AdS_3 \times S^3 \times K3$ .

In many aspects the *AdS/CFT* duality in three dimensions is special. At first gravity in the  $AdS_3$  background is a topological theory with a two-dimensional CFT on its boundary. Therefore, all dynamical degrees of freedom have to originate from the compact part of spacetime. The purely topological nature of the theory makes quantisation practicable [5,6,7], and it is the only example showing that the KK-spectrum and the modes of the conformal field theory coincide. It has been Vafa [8], who first observed that the chiral primaries and its descendants of the CFT are not enough to account for all states in the KK spectrum. Later, this puzzle has been solved by observing that some of the multi-particle states from supergravity correspond to non-chiral primaries, which itself are not descendants from any other chiral fields but the product of descendants of chiral primaries [9,10]. And although the missing states can be constructed this way, they are not part of the original CFT spectrum as would have been expected by the *AdS/CFT* duality. To solve this problem we propose a correspondence between *AdS* and a supersymmetric  $W_\infty(\lambda)$  algebra.

To get a better understanding how W algebras enter the discussion, we will summarise the main ideas as follows. Take for example  $Q_5$  D5-branes and  $Q_1$  D1-branes wrapped on  $X$ , leaving a string in the remaining six dimensions. As long as no further fields are turned on, the branes can freely join and separate. The corresponding effective 1+1 dimensional field theory of this system contains a  $SU(Q_5)$  gauge group with  $Q_1$  additional instantons. The two extremal constellations, with all branes separated or joined, translate to the Coulomb respectively Higgs branch of the low energy theory. This passage between two sectors with all its intermediate states are the “dynamical” degrees of freedom. But moving

branes in a curved background are not well understood because of lack of a good description for the low energy sector. This is different for the moduli space  $\mathcal{M}_{(Q_5, Q_1)}$  which is expected to be equivalent to a  $(4, 4)$  sigma model on the target space  $\text{Sym}^N(X)$  for  $N = Q_5 Q_1$  [7, 11]. But this description has two major drawbacks. First,  $N$  has to be large and  $Q_5$  and  $Q_1$  have to be relatively prime, otherwise the moduli space is reducible and the representation as a symmetric product is not valid, and second, the moduli space depends on the product of the charges only, which contradicts the effective field theory description. Both problems have a solution by forming the infinite sum

$$\mathcal{M} = \sum_{N=1}^{\infty} \coprod_{\nu: \text{partition of } N} [X]^{\nu}. \quad (1.1)$$

Similar to the calculation of the Poincaré polynomials of Hilbert schemes by Göttsche's formula, the infinite sum over all partitions of  $N$  leads to important simplifications. We will show that, if the cohomology classes of  $X$  can be represented by a chiral ring in the topological SCFT, the cohomology classes of  $\mathcal{M}$  are accessible by the corresponding supersymmetric  $W_{\infty}(1/4)$  algebra.

How does this construction solve the problem of the missing states? The first nontrivial example is the distribution of three D1-branes in an infinite stack of D5-branes, each of charge one. Because only the product  $Q_5 Q_1$  of charges enters the construction, it is always possible to embed a finite system of D1-branes into an infinite stack of D5-branes. This way there are only three different partitions for  $N = 3$

$$[3, 0, 0, \dots], \quad [2, 1, 0, \dots] \quad \text{and} \quad [1, 1, 1, \dots]. \quad (1.2)$$

The first and the third vector resemble primary states of the original SCFT and are related by T duality. They correspond to the Higgs respectively the Coulomb branch of the low energy description and have been studied in [6, 7]. The coproduct in (1.1) reduces to a simple product and the moduli space factorises which leaves a Liouville theory as SCFT. The second partition in (1.2), however, represents one of the missing states. In analogy to representation theory, the previous problem of classifying the physical states on the CFT side, splits into two parts. The first uses representation theory of the symmetric group, as seen above. Whereas the second part relies on the topology of  $X$  only and not on that of its symmetric products. This assertion needs further explanation.

Following the previous example, consider the primary state corresponding to the partition  $[1, 1, 1, \dots]$ . Now it is a classical result from rational conformal field theory, that

its null states correspond to Schur polynomials which, on the other hand, are representations of the symmetric group. These special polynomials can be generated by repeated operation with a differential operator on the lower lying partitions. With reference to the above example, the state  $[2, 1, 0, \dots]$  is the result of this operator acting on  $[1, 1, 0, \dots]$ . Of course, there is no difference whether one is calculating the null state from the Liouville theory and then operating with the shift operator or one is first acting with the operator on the Liouville Lagrangian and then constructing the null state. As we will show, the second possibility has a general solution in form of a  $W_\infty(\lambda)$  algebra, which will be the main result of this article. In this setting, the duality between a supersymmetric  $AdS$  background and a conformal field theory is completely determined by the Liouville action which is one reason why the original  $AdS/CFT$  correspondence works so well. Making use of results from Hilbert schemes of singular points [12,13,14,15], we will show that the only topological information entering the conformal field theory are the Euler characteristic and the canonical class of  $X$ . Especially the article [15] by M. Lehn has been very important for understanding the mathematical aspects of the construction.

This paper is organised as follows. In Section 2, we review the conformal field theory of the untwisted sector as introduced in [6,7] and explain the puzzle of the missing states for the twisted sector [8,9]. Section 3 deals with the distribution of D1-branes on a stack of D5-branes and its representation as Young diagrams. We compare these to the null states of the previous section and review the relation to Schur, respectively Jack polynomials. The purpose of Section 4 is to review the mathematical definition of moduli spaces and its representation as schemes. From the lectures of Nakajima [12] we present two different formulations, one as a field theory and one in terms of a Virasoro algebra. The W algebra is constructed in Section 5 and possible generalisations are considered. The final Section 6 contains the conclusions and further suggestions for future investigation.

## 2. String Theory on $AdS_3 \times S^3 \times X$

In this section we review the compactification of string theory in the  $AdS_{D+1} \times \mathcal{N}$  background. Here  $\mathcal{N}$  will be a compact manifold whose holonomy group is large enough for at least some supersymmetry to survive. According to the work of Seiberg and Witten [7] only the case  $D = 2$  is in general stable under quantisation. In the following it will be therefore necessary to consider the conformal field descriptions of  $AdS_3$  along the lines of [5,6] as well as the reduction to the large brane [7]. The first representation is appropriate for demonstrating the problem of the missing states in the CFT description [8,9,10], whereas the second version is necessary for the construction of the W algebra in Section 5.

## 2.1. Classical Results

In [2] Maldacena proposed a remarkable correspondence between a fixed supergravity background  $AdS_{D+1} \times \mathcal{N}$  and a supersymmetric conformal field theory depending on the dimension  $D$  and  $\mathcal{N}$ . The mapping between these two descriptions includes a stack of  $N$  black D-1 branes. From a more topological point of view, these branes carry a gravitational instanton <sup>2</sup> of charge  $N$ . Basically there is no great difference between a classical gravitational instanton and a gauge instanton in Euclidean space. In both cases, the charge of the topological solution acts as an additional coordinate in the moduli space, and although the moduli space has no natural metric, it carries a conformal structure which is often sufficient to determine its topology. The *AdS/CFT* correspondence now states that, in the limit  $N \rightarrow \infty$ , both moduli spaces coincide or stated differently, their boundaries meet at this point. That it is exactly one point, can be seen from the example stated in the introduction. It is not important where the D1-branes are located on the D5-brane for the limit  $Q_5 \rightarrow \infty$ , which is the reason why the target space of the sigma model depends on the product  $Q_5 Q_1$  only. Using the picture of Maldacena, the topological properties of gravitational instantons are mapped to the boundary of  $AdS_{D+1}$  with additional Chern-Simons terms located at  $N$  arbitrary points on its boundary.

The hope is that not only the two moduli spaces join each other along the boundary, but that it is actually possible to move from one region to the other in a definite way. A similar phenomenon is known from mirror symmetry, where the Kähler and the complex structure guarantee for a meromorphic moduli space in the large complex structure limit, but even in this case it is not sure that the two moduli spaces intersect in more than one point. But, if one accepts the idea that there is a description of quantised supergravity in terms of gauge instantons, there has to be a mapping for all finite brane configurations and not only in the limit  $N \rightarrow \infty$ . Because the moduli space of gauge instantons is conformally invariant, we have to assume the same on the side of supergravity. So, take a subset  $W \subset AdS_{D+1}$  with the restriction that the boundary  $\partial W$  carries a conformal structure. This is exactly the case studied in [7], with  $W$  large and  $\partial W$  near the boundary of  $AdS_{D+1}$ . Although it is not evident, that this is a valid description of the underlying moduli space, it still reproduces the results as expected from the field theory approximation.

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<sup>2</sup> Here and in the following we will understand a topological configuration as an instanton.

In contrast to the microscopic formulation of the D-brane modes in the *AdS/CFT* correspondence the large brane can be analysed by classical calculations. The first important information one gets in this limit is the underlying structure of the moduli space and its stability, which depends strongly on the dimension of the *AdS* space and the signature of the scalar curvature. As shown in [7], the boundary of *AdS*<sub>*D*+1</sub> has a natural conformal structure but no natural metric. Thus the classical form

$$ds^2 = r_0^2(dr^2 + \sinh^2 r \, d\Omega^2) \quad (2.1)$$

only gives rise to a boundary with a metric of gauged conformal group. Because this gauge fixes the scalar curvature at the boundary, it eliminates the freedom to deform the theory to different boundaries and thus to different string vacua. It is thus not possible to study the stability of the moduli space. The problem can be solved by a simple reparametrisation of the radial coordinate. Instead of the metric (2.1) with its fixed spherical boundary, it is easier to begin with an arbitrary metric  $ds^2 = g_{ij}dx^i dx^j$  on  $\partial W$  of fixed conformal structure. This metric has an unique embedding into *W* by

$$ds^2 = \frac{r_0^2}{t^2} (dt^2 + \hat{g}_{ij}(x, t) dx^i dx^j) \, , \quad (2.2)$$

with the boundary condition

$$\hat{g}_{ij}(x, 0) = g_{ij}(x) \, . \quad (2.3)$$

A Taylor expansion in the variable *t* near the boundary relates the conformal parameter to the radial coordinate of (2.1) by  $t = 2e^{-r}$ . Standard *AdS/CFT* calculations then show that the variable *t* is related to the physical scalar field  $\phi$  by the scaling relation  $\phi \sim t^{-(D-2)/2}$  for  $D > 2$  and a logarithmic dependence for  $D = 2$ , which is typical for a Liouville field. The final relation between the radial coordinate *r* and the physical relevant field  $\phi$  can then be summarised by

$$r = \begin{cases} \frac{2}{D-2} \log \phi + \frac{1}{(D-1)(D-2)} \phi^{-\frac{4}{D-2}} R & \text{for } D > 2 \\ \phi + e^{-2\phi} \phi R & \text{for } D = 2 \end{cases} \quad (2.4)$$

Obviously the two-dimensional case behaves quite differently and one could argue that this is only a pathological case. But later in this section we will show how  $\phi$  is related to the eleventh dimension in M theory, and thus plays a fundamental role in many brane interactions.

Now that the metric and its physical field content are known, the classical Lagrangian of a BPS saturated large brane in  $D$  dimensions can be calculated from the DBI and WZW action [7]

$$S = \begin{cases} \frac{Tr_0^D}{2^{D-3}(D-2)^2} \int \sqrt{g} \left( (\partial\phi)^2 + \frac{D-2}{4(D-1)} \phi^2 R + \mathcal{O}(\phi^{\frac{2(D-4)}{D-2}}) \right) & \text{for } D > 2; \\ \frac{Tr_0^2}{2} \int \sqrt{g} \left( (\partial\phi)^2 + \phi R - \frac{1}{2} R + \mathcal{O}(e^{-2\phi}) \right) & \text{for } D = 2. \end{cases} \quad (2.5)$$

Because we started with an Euclidean version of  $AdS$ , the integration is over the compact space  $S^D$  respectively  $\partial W$ . An analogous discussion for the Minkowski space should be possible but its results for the moduli space is not clear to us, so that we will omit this point.

What is the main difference between the two-dimensional boundary and the case  $D > 2$ ? For  $\phi$  large and constant the integrand of (2.5) reduces to the potential terms  $\phi^2 R$  and  $\phi R$ . It is not very surprising that the sign and thus the stability of a physical state of the dominant part of the field theory description depends only on the scalar curvature of the conformal space  $\partial W$ . For  $D > 2$ , the BPS saturated D-branes are free to move relatively to each other as long as only one type of brane is involved. This picture changes of course if additional branes are included. But we will show that in the moduli space a sector of Liouville type develops, which is of the same type as the one found for  $D = 2$ . The basic information we get for the moduli space of  $D > 2$  thus is that a monoculture of D-branes has a rather trivial moduli space, and we believe that the  $AdS/CFT$  correspondence is actually exact. One example is the case of a stack of black D3-branes in an  $AdS_5 \times S^5$  background. The dual conformal field theory is  $\mathcal{N} = 4$  supersymmetric in four dimensions with  $SU(N)$  gauge theory. The only contribution to the otherwise trivial beta function are instanton corrections. This is a strong evidence that the moduli space is basically the moduli space of  $SU(N)$  gauge instantons.

Things change dramatically in the case of  $D = 2$  or nontrivial brane interactions. The  $\phi R$  part in (2.5) is not only a potential on the D-brane action, but a topological term, proportional to the conformal charge in the Liouville action. Furthermore the moduli space develops an infinite tube if a D1-brane instanton shrinks to zero size and separates from the D5-brane, dividing the Coulomb from the Higgs branch [16]. But, if this picture is correct, the opposite version has to be valid, too, namely that the moduli space of any D-brane configuration can be constructed from two basic elements, the Coulomb and the Higgs branch of one D1-instanton only. This is exactly what we will do in Section 5.

## 2.2. The Liouville Theory

Here we will study the Higgs branch of the Liouville theory in the background of one D1-brane. As stated above, the understanding of this sector is the first step in constructing moduli spaces of intersecting branes.

Before entering the construction of the two-dimensional conformal field theory, some details concerning the configuration of the branes are necessary. The D5-branes are located in the  $(x_0, x_1, x_6, x_7, x_8, x_9)$  plane of the ten-dimensional space, whereas the D1-branes are stretched in the  $x_0, x_1$  directions. The last four dimensions of the D5-brane are wrapped on a manifold  $X$ , where  $X$  is either  $T^4$  or  $K3$ . Somewhere on this compact space  $X$  each D1-brane is fixed in one point, and to simplify our discussion we will assume that all these points are located along the  $x_6$  coordinate. To complete the brane spectrum for the type IIB string theory we insert D3-branes in the  $(x_0, x_1, x_8, x_9)$  plane. This choice of coordinates suggests that the D3-branes are unaffected by the introduction of the D1-branes which is definitely not true. But as a starting point it simplifies the construction of the corresponding CFT considerably. And as further motivation, one can imagine that the set of fixed points on  $X$  are free to move and thus can be located at the coordinates  $x_6, x_7$ . In terms of the CFT on  $X$ , the dynamics of the D-branes decouple, but later on the vanishing of the total conformal charge intertwine the degrees of freedom and generates an interaction between the different types of branes.

In analogy to the considerations in [17,18], the D-brane configuration can be interpreted in terms of M theory. For this we enlarge the ten-dimensional string description by the variable  $x_{10}$ , compactified on a sphere  $S^1$  of radius  $R$ . The coordinate  $t$  of the metric (2.2) can then be identified as the complex moduli parameter

$$t = 2 \exp(-(x_6 + ix_{10})/R) \quad (2.6)$$

from the eleventh dimension, whereas the additional parameter of the D3-branes

$$v = x_8 + ix_9 \quad (2.7)$$

has no entry up to now. The moduli parameters  $(t, v)$  specify the positions of the branes as the roots of a polynomial  $F(t, v)$ . The interpretation of  $F$  as an algebraic curve and its connections to gauge instantons is well understood [17,18], but its connection to the *AdS/CFT* correspondence allows a more direct interpretation from the viewpoint of M theory. Take for example the string limit  $R \rightarrow 0$ . From the definition (2.6) we see that



$t$  vanishes and the metric (2.2) reduces to the boundary of  $W$ , while the number of D5-branes goes to infinity. Therefore, the complex “radius” of  $AdS$  has to be identified with  $r = (x_6 + ix_{10})/R$ , which is proportional to  $N = Q_5 Q_1$ . At first view the complex value of  $r$  for  $x_6 \neq 0$  may seem to contradict our identification, but actually it is not  $r$  we have to compare but the complex field  $\phi$ , or stated differently, one needs at least one D1-brane to ensure a complex field. What is the interpretation of the moduli parameter (2.7) in the context of  $AdS/CFT$ ? The sources of D3-brane charges are orbifold singularities on  $W$ . One explicit example for  $AdS_5 \times S^5$  with an analysis of the moduli space can be found in [19]. Thus  $v$  parametrises the deformation of the conformal metric  $\widehat{g}_{ij}(x, t; v)$  in (2.2). Because the interpretation of D3-branes along the lines of [6] as an orbifold on  $\partial W$  is not very intuitive. This is why we leave it to Section 5 to give a further analysis along the lines of Hilbert schemes.

In the following we review the conformal field theory on  $AdS_3 \times S^3 \times T^4$  near the boundary of  $AdS_3$ . In this limit the metric (2.1) reduces to

$$ds^2 = dr^2 + e^{2r} \partial\gamma \partial\bar{\gamma} . \quad (2.8)$$

After the analytic continuation of the radial coordinate, the worldsheet Lagrangian is

$$\mathcal{L} = \bar{\partial}\phi\partial\phi + e^{2\phi}\bar{\partial}\gamma\partial\bar{\gamma} , \quad (2.9)$$

and can be put into the standard form of a Liouville action after introducing an auxiliary field  $\beta$  and the improvement term from (2.5)

$$\mathcal{L} = \bar{\partial}\phi\partial\phi - \frac{2}{\alpha_+} \widehat{R}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \beta\bar{\beta} \exp\left(-\frac{2}{\alpha_+}\phi\right) , \quad (2.10)$$

with the Liouville parameter  $\alpha_+^2 = 2k - 4$ . In the near horizon limit, the conformal field theory description on  $AdS_3 \times S^3 \times T^4$  factorises into three separate WZW models. The  $AdS_3$  part gives rise to an affine  $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$  left-right symmetric group manifold at level  $k > 2$ , which again determines the conformal charge of the  $S^3$  theory on the group manifold  $SU(2)$ . In this section we chose  $X$  to be  $T^4$ , because of its simple representation as an Abelian  $U(1)^4$  model. Later on, the case of  $K3$  will be more appropriate because of its simple cohomological structure, where the additional  $\mathbf{Z}_2$  orbifolding has no effect on our reasoning.

Near the horizon of  $AdS_3$  there are two alternative descriptions. The RNS formulation has its advantage in the calculation of the particle spectrum, whereas the quantisation of

the Liouville Lagrangian (2.10) in the Green-Schwarz representation gives a better understanding of the sigma model on  $T^4$ . But in both cases the algebra ends up to describe a  $(4, 4)$  supersymmetric CFT. In the WZW description, the fermions and the bosonic part of the currents of  $AdS_3$  are denoted by  $(\psi^A, k^A)$ , while those corresponding to  $SU(2)$  are  $(\chi^a, j^a)$ . Here we adapted the notation from [6] to those of [7] for the sake of clarity, although the OPEs are not completely identical. Now, the construction of the supersymmetric conformal algebra is straightforward. From the complete algebra [6], the parts we will need are only the contributions of the energy-momentum tensor and the supercurrent for  $AdS_3 \times S^3$ . In the RNS formulation, the SCFT of  $T^4$  is a free field contribution of conformal charge  $c = 6$  and denoted  $T_{T^4}$  respectively  $G_{T^4}$ . With these simplifications, the  $\mathcal{N} = 4$  algebra reduces to

$$\begin{aligned} T_X &= \frac{1}{Q_5} (k^A k_A - \psi^A \partial \psi_A) + \frac{1}{Q_5} (j^a j_a - \chi^a \partial \chi_a) + T_{T^4} \\ G_X &= \frac{2}{Q_5} \left( \psi^A k_A - \frac{i}{3Q_5} \epsilon_{ABC} \psi^A \psi^B \psi^C \right) + \frac{2}{Q_5} \left( \chi^a j_a - \frac{i}{3Q_5} \epsilon_{abc} \chi^a \chi^b \chi^c \right) + G_{T^4} . \end{aligned} \quad (2.11)$$

The analogous formulation in the Green-Schwarz description follows from (2.5), where the four spinor fields on  $AdS_3 \times S^3$  are denoted by  $S^\mu$  in addition to the Liouville field  $\phi$ .

$$\begin{aligned} T_\phi &= -\frac{1}{2} \partial S^\mu S_\mu - \frac{1}{Q_5} j^a j_a - \frac{1}{2} \partial \phi \partial \phi + \frac{1}{\sqrt{2}} \left( \sqrt{Q_5} - \frac{1}{\sqrt{Q_5}} \right) \partial^2 \phi + T_{T^4} \\ G_\phi^\mu &= \frac{1}{\sqrt{2}} \partial \phi S^\mu - \frac{2}{\sqrt{Q_5}} \eta_{\mu\nu}^a j_a S^\nu + \frac{1}{6\sqrt{Q_5}} \epsilon_{\mu\nu\sigma\rho} S^\nu S^\sigma S^\rho - \left( \sqrt{Q_5} - \frac{1}{\sqrt{Q_5}} \right) \partial S^\mu + G_{T^4} . \end{aligned} \quad (2.12)$$

In both cases the conformal charge is  $c = 6Q_5$  and corresponds to the case of  $Q_5$  D5-branes with one D1-brane on its surface. This is of course not very satisfying, because it fixes the possible number of D1-branes. Taking a closer look at the Liouville part of (2.12), the improvement term of the energy-momentum tensor suggests a generalisation for  $Q_1$  D1-branes in form of a minimal conformal model

$$\frac{1}{2} \partial \phi \partial \phi - Q \partial^2 \phi \quad (2.13)$$

with Liouville charge

$$Q = \frac{1}{\sqrt{2}} \left( \sqrt{\beta} - \sqrt{1/\beta} \right) , \quad (2.14)$$

for  $\beta = Q_5/Q_1$ . But, as we will demonstrate in Section 3, even if such an extension exists, there would still be not enough KK states to get a one-to-one mapping with the primary modes and its descendants. To get a better understanding of this problem one has to take a closer look at the higher twist modes for large  $N = Q_5 Q_1$ .

### 2.3. Missing States

It was Vafa [8] who first pointed out a discrepancy between the number of KK modes in the  $AdS_3 \times S^3$  supergravity background and the number of chiral primaries and their descendants. An observation, which was further analysed by de Boer [9], who suggested a solution [10] by considering the “exclusion principle” as first observed in [5]. But, although it is very reassuring to know that the particle spectrum for both sides of the  $AdS/CFT$  description coincide in principle, one has to ask to what states the original missing states translate in the (2.12) description. To answer this question, the first step is the understanding of the KK modes in the context of representation theory.

What makes the quantisation of  $AdS_3 \times S^3$  so easy, is the description of both spaces as group manifolds. In Subsection 2.1 we started by the formulation of the conformal field theory on the group manifolds  $SL(2, \mathbf{R})$ ,  $SU(2)$  and guaranteed a consistent supersymmetric formulation by the vanishing conformal charge and a further GSO projection. And, because the worldsheet of the two-dimensional CFT is a cylinder, the left-right modes decouple so that it was possible to simply ignore the left moving part of the modes. Things are similar for the  $AdS$  description, but with the supersymmetric analog of  $SL(2, \mathbf{R})$ . The spherical harmonics of  $S^3$  are representations of  $SO(4)/SO(3)$  or in the more appropriate form  $SU(2) \times SU(2)/SU(2)$ , whereas the  $AdS_3$  space decomposes into the left-right symmetric group  $SU(2|1, 1) \times SU(2|1, 1)$ . The representation of short and long multiplets transforming under this group can be found in [20] and is reviewed in [9]. Now, the KK modes have a complete description as short multiplet representations of  $SU(2|1, 1)$ , decomposed under the diagonal group  $SU(2)$ .

The short multiplets are of special importance as they contain the massless spin 2 fields of the supergravity theory. And, because this is the highest possible spin by KK compactification, the multiplication of these short multiplets has a massless single particle state as highest spin state. For this reason, the complete set of chiral primaries of the CFT is obtained by taking an arbitrary product of single particle states on the supergravity side. This is exactly the case covered by (2.12). But of more interest to us is the multiparticle spectrum which caused the original puzzle of the missing states [8]. These have been identified in [9] as non-chiral primaries and are thus elements of long multiplets. The basic observation was that the tensor product of two short multiplets does not contain a new short multiplet only but various longer ones as well. In the next section, we will analyse the structure and origin of the multiparticle spectrum from the conformal field description and the supergravity point of view.

### 3. The Partition of Branes

In this section we pursue the analysis of the multiparticle spectrum one step further. A simple argument shows, that there is no generalisation of (2.12) to D1-brane charges larger than one. But, nonetheless, a comparison between the particle spectrum of the CFT and the supergravity theory reveals the structure of the missing states and allows their general construction in terms of Jack polynomials.

#### 3.1. Null States and Jack Polynomials

In Section 2 we reviewed the effective action of the long string in an  $AdS_3 \times S^3 \times X$  background. The basic structure, which governs the residual supersymmetric algebra, is a Liouville term of the form (2.13) with improvement term (2.14). For  $Q_1 = 1$  this reduces to the low energy description (2.12) with the coupling constants [7]

$$\begin{aligned} g_C(\phi) &= \exp\left(\frac{1}{\sqrt{2}} \frac{-1}{\sqrt{\beta}} \phi\right) \\ g_H(\phi) &= \exp\left(\frac{1}{\sqrt{2}} \left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right) \phi\right) \end{aligned} \quad (3.1)$$

for the short string of the Coulomb branch and respectively the large string of the Higgs branch. Because the coupling constants enter the vertex operators as additional screening charges, they characterise the vacuum of the Liouville theory. In the following, we will show that the generalisation to arbitrary values of  $\beta = Q_5/Q_1$  is wrong. To demonstrate the failure of this description we take the mode expansion of (2.13) in the free field representation

$$\mathcal{L}_n = \frac{1}{2} \sum_{n \in \mathbf{Z}} a_{n+m} a_{-m} - \alpha_0 (n+1) a_n \quad (3.2)$$

with  $\alpha_0 = \frac{1}{\sqrt{2}}(\sqrt{\beta} - \sqrt{1/\beta})$  and conformal charge  $c = 1 - 12\alpha_0^2$ . Instead of writing down the explicit form of the screened vertex operators, it is sufficient to consider the highest weight states of the Fock vacuum, represented by  $|\alpha_{r,s}\rangle$  with

$$\alpha_{r,s} = \frac{1}{\sqrt{2}} \left( (r+1)\sqrt{\beta} - (s+1)\sqrt{1/\beta} \right). \quad (3.3)$$

The values of  $r$  and  $s$  parametrise the partition of D1-branes on the stack of D5-branes restricted to

$$1 \leq s \leq Q_5 - 1 \quad \text{and} \quad 1 \leq r \leq Q_1 - 1 \quad (3.4)$$

for  $\beta = Q_5/Q_1$ . To compare these vacuum states with the results from (3.1) one has to take into account that the addition of the four spin fields  $S^\mu$  and the currents  $j^a$  in the algebra (2.12) results in a shift in  $s$  and  $r$  by two units. Therefore, it is useful to interpret the additional screening charges for the string vertex operators (3.1) in the framework of the Liouville theory as the vacuum states at  $s = 0$  and  $s = Q_5$  for fixed  $r = 0$ . But to keep things simple, we will stick to the range (3.4) for the values of  $r$  and  $s$  and keep in mind that the interpretation as string couplings of (2.12) is related to the Liouville vacuum state by a shift in the parameters. With this agreement, the Higgs branch of the long string has the simple representation  $|\alpha_{0,0}\rangle$  and the short string of the Coulomb branch corresponds to  $|\alpha_{0,Q_5}\rangle$ . Having the KK particle spectrum identified with the Liouville states of (2.13), the origin of the missing states becomes more clear.

The generalisation of the states  $|\alpha_{0,s}\rangle$  along (3.1) is obvious. But what about the other highest weight states  $|\alpha_{r,s}\rangle$ , defined by the relation  $a_0|\alpha_{r,s}\rangle = \alpha_{r,s}|\alpha_{r,s}\rangle$ ? The reducible vacuum states of the minimal model decompose under the null vectors, of which the first four take the form

$$\begin{aligned}
|\chi_{1,1}\rangle &\sim a_{-1}|\alpha_{1,1}\rangle \\
|\chi_{1,2}\rangle &\sim \frac{1}{2} \left( a_{-2} + 2\sqrt{\frac{\beta}{2}}a_{-1}^2 \right) |\alpha_{1,2}\rangle \\
|\chi_{2,1}\rangle &\sim \frac{1}{2} \left( a_{-2} - \sqrt{\frac{2}{\beta}}a_{-1}^2 \right) |\alpha_{2,1}\rangle \\
|\chi_{1,3}\rangle &\sim \frac{1}{3} \left( a_{-3} + 3\sqrt{\frac{\beta}{2}}a_{-2}a_{-1} + \beta a_{-1}^3 \right) |\alpha_{1,3}\rangle \\
|\chi_{3,1}\rangle &\sim \frac{1}{3} \left( a_{-3} - \frac{3}{2}\sqrt{\frac{2}{\beta}}a_{-2}a_{-1} + \frac{1}{\beta}a_{-1}^3 \right) |\alpha_{3,1}\rangle .
\end{aligned} \tag{3.5}$$

Here we omitted the normalisation factors which will be of no importance in our discussion. The vacuum states  $|\alpha_{r,s}\rangle$  are invariant under two symmetries, what reduce the infinite number of possible states to a finite set. A shift in the parameters  $r$  and  $s$  of the vacuum state by  $|\alpha_{r-Q_1,s-Q_5}\rangle$  has no effect and thus justifies the reduction of the parameters to the range (3.4). So the highest possible spin for  $N = Q_5Q_1$  is exactly  $N$  and at the same time the origin of the stringy exclusion principle as found in [5]. The second symmetry exchanges the parameter  $\beta \rightarrow 1/\beta$  with an additional sign change for all operator modes  $a_n \rightarrow -a_n$ . From a string theory point of view, this inversion corresponds to T-duality along the four compact dimensions of  $X$ . Up to now, we have assumed that the number

of D-branes  $Q_5$  and  $Q_1$  should be prime, so that the quotient  $\beta$  would be in one-to-one correspondence with their charges. But from now on we will drop this condition and analyse, as a first step, the null states as functions of  $\beta$ .

The classical moduli space for  $Q_5$  and  $Q_1$  is simply the symmetric product  $X^{[N]}$ . But the case that the two charges are not prime contradicts the Liouville structure of (2.5) and its representation within the Green-Schwarz formulation. To be more specific, take the example  $Q_5 = b Q_1$  or  $\beta = b$ , with the largest possible conformal charge  $c = 1$  obtained for  $b = 1$ . At this value the moduli space degenerates and the Liouville theory reduces to the free field representation of the Virasoro algebra. The first null state, not defined for  $\beta = 1$  is  $|\chi_{2,2} \rangle$ . Another interesting value is  $\beta = 2$ , for which the null states (3.5) reduce to Schur polynomials. These functions enter the representation theory as the polynomial ring of the symmetric group  $S_N$  and thus generate a basis for the moduli space  $X^{[N]}$ . To clarify the origin of the missing states in this new framework, we will give a short introduction to these Schur polynomials and its generalisation to Jack polynomials for arbitrary  $\beta$ . Here and in the following, we will use the conventions of the review [12] where further references can be found.

A partition  $\lambda = [\lambda_1, \lambda_2, \lambda_3, \dots]$  is a nondecreasing sequence of nonnegative integers for a finite number of  $\lambda_i \neq 0$ . A different way of presentation is  $\lambda = (1^{m_1}, 2^{m_2}, \dots)$  with  $m_k = \#\{i | \lambda_i = k\}$ . The two notations are distinguished by the different type of brackets. Characteristic numbers of a partition are the sum of integers  $\lambda_i$ , denoted by  $|\lambda| = N$  and the number of nonzero entries in  $\lambda$ , noted as the degree and the length  $l(\lambda)$  of a partition. It is often useful to give the vector  $[\lambda_1, \lambda_2, \lambda_3, \dots]$  a graphical interpretation as Young diagrams, but we will not make use of them here. The ring of symmetric functions with rational coefficients is denoted

$$\Lambda_N = \mathbf{Q}[x_1, \dots, x_N]/S_N, \quad (3.6)$$

where the symmetric group  $S_N$  acts by permutation on the variables. As already mentioned in the introduction, it is always possible to embed a partition in the infinite dimensional space  $\Lambda_\infty$ . The most general representative of  $\Lambda_\infty$  is the monomial symmetric function or orbit sum

$$m_\nu = \sum_{\text{dist.perm.}} x_1^{\alpha_1} \cdots x_N^{\alpha_N}, \quad (3.7)$$

where the sum is over all distinct permutations  $\alpha = [\alpha_1, \alpha_2, \dots] \leq \nu$  of entries in  $\nu$  with  $l(\alpha) \leq N$ . There are two distinguished partitions for each integer  $n$ , the elementary

symmetric function  $e_n = m_{(1^n)}$  with  $l(\alpha) = n$  and the power sum  $p_n = m_{(n)}$  with  $l(\alpha) = 1$ . Because of the ring structure it is possible to represent the functions  $e_n$  and  $m_\nu$  by the power sum  $p_n$ , with the monomial symmetric functions recursively expressed in terms of  $p_n$  by

$$p_i m_\nu = \sum_{\mu} a_{\nu\mu} m_\mu , \quad (3.8)$$

where the summation runs over the partitions of  $|\nu| + i$  and the coefficients  $a_{\nu\mu}$  counting the number of multiplicities of entries in  $\mu$ . The elementary symmetric functions are expressed more easily by the generating function

$$E(z) = \sum_{n=0}^{\infty} e_n z^n = \prod_{i=1}^{\infty} (1 + x_i z) . \quad (3.9)$$

At  $z = 1$  this has the structure of the tree-level amplitude of a single chiral fermion [12,21]. For completeness, we will give the corresponding bosonic amplitude, too

$$H(z) = \sum_{n=0}^{\infty} h_n z^n = \prod_{i=1}^{\infty} \frac{1}{(1 - x_i z)} , \quad (3.10)$$

with the complete symmetric functions  $h_n$ . They are related to  $e_n$  by  $E(z)H(-z) = 1$  and are complementary to each other. An alternative representation in terms of  $p_n$  is given by

$$H(z) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} p_n z^n\right) \quad \text{and} \quad E(z) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} p_n z^n\right) . \quad (3.11)$$

The generating functions have a structure similar to a scalar bosonic field

$$\phi = \sum_{n=-\infty}^{\infty} \frac{1}{n} a_n z^n , \quad (3.12)$$

with a zero mode  $a_0$  still to be defined. We postpone this calculation to Section 5, after a more detailed analysis of the moduli space. With a redefinition of  $z \rightarrow z/n$ , the function  $H(z)$  becomes the generating function of the Schur polynomials of which the first representatives are  $P_0(x) = 1$ ,  $P_1(x) = x_1$ ,  $P_2(x) = x_2 + (1/2)x_1^2$  and  $P_3(x) = x_3 + x_2x_1 + (1/6)x_1^3$ . Comparing these polynomials with the null vectors  $|\chi_{r,1}\rangle$  shows, that the vacuum states (3.5) correspond to partitions of the type  $\lambda = (r^s)$  only. But already for  $N = 3$  this

representation is not sufficient to represent all possible partitions. From the example in [12] we know the three different orbit sums for  $N = 3$

$$\begin{aligned} m_{[1,1,1]} &= \frac{1}{3}a_{-3} - \frac{1}{2}a_{-2}a_{-1} + \frac{1}{6}a_{-1}^3 \\ m_{[2,1,0]} &= -a_{-3} + a_{-2}a_{-1} \\ m_{[3,0,0]} &= a_{-3} \end{aligned} \tag{3.13}$$

of which only the first one has a counterpart in (3.5). The other two partitions are Jack polynomials  $J_\lambda(x; \beta)$ . For the positive and real number  $\beta$  we define an inner product in the ring of symmetric functions by

$$\langle p_\lambda, p_\mu \rangle = \beta^{l(\lambda)} z_\lambda \delta_{\lambda\mu}, \tag{3.14}$$

where  $z_\lambda = \prod k^{m_k} m_k!$  for the partition  $\lambda = (1^{m_1}, 2^{m_2}, \dots)$ . With this normalisation, the Jack polynomials are defined recursively by the Gram-Schmidt method

$$\begin{aligned} J_\lambda(x; \beta) &= \sum_{\mu \leq \lambda} u_{\lambda\mu}(\beta) m_\mu(x) \\ u_{\lambda\lambda}(\beta) &= c_\lambda(\beta) \quad \text{and} \quad \langle J_\lambda(x; \beta), J_\mu(x; \beta) \rangle = 0 \quad \text{if} \quad \lambda \neq \mu \end{aligned} \tag{3.15}$$

with the normalisation factor

$$c_\lambda(\beta) = \prod_{s \in \lambda} (\beta a(s) + l(s) + 1), \tag{3.16}$$

where the vector  $s \in \lambda$  is a point in the Young diagram with  $a(s)$  arms and  $l(s)$  legs. The appearance of Jack polynomials is common for algebraic integrable systems as eigenvalues of the Hamiltonian function. For the case at hand, the system of null states resembles the Calogero-Moser model, as already noted in [13], where this information has been used to compute the Virasoro algebra for the homology of the Hilbert scheme. Before we pick up this aspect in Section 4, let us mention one further characteristic of the Jack polynomials, which will prove important for the construction of the W algebras.

Besides from the Hamiltonian, the Jack polynomials have a further symmetry, generated by the Dunkl operators [22]

$$D_i = \beta x_i \frac{\partial}{\partial x_i} + \sum_{i \neq j} \frac{x_i}{x_i - x_j} (1 - K_{ij}), \tag{3.17}$$



for  $i = 1, \dots, l(\lambda)$  and the matrix  $K_{ij}$  interchanging the positions of  $x_i$  and  $x_j$ . One interesting property of this operator has been analysed in [22], where  $D_i$  has been used to construct creation operators  $B_k^+$ , whose action onto the trivial partition generate Jack polynomials, as each operator adds one further column of length  $k$  to the Young diagram and thus has the property of shifting each line by one step. In Section 5 we will construct similar creation operators to generate the missing partitions of (3.13) from primary states. The successive action of these shift operators takes us to the W algebras.

### 3.2. The Twisted States

Having a hand on all possible partitions, we still have to show that the missing states in the KK spectrum have a representation as elements of the just introduced polynomials. For this we go back to an example given in [9] and relate the non-chiral primaries with cohomology classes of the moduli space. The possible products of the differential forms then have a simple representation as Young diagrams. It is of no surprise that all these states belong to the twisted sector and thus did not enter the previous discussion.

In the introduction we mentioned the relation of matrix theory and compactification on the  $AdS_3 \times S^3$  background. The advantage of this formulation in the low energy limit is its immediate interpretation as field theory degrees of freedom in a six-dimensional spacetime [3,23]. The description of a stack of  $N = Q_5 Q_1$  D-branes in the Higgs branch takes the form of a field  $X^\mu$  with  $\mu = 1, \dots, 6$ , which can be written as a diagonal matrix

$$X^\mu = U x^\mu U^{-1}, \quad U \in U(N) \quad (3.18)$$

after a  $U(N)$  rotation. The eigenvalues of  $x^\mu$  are then the coordinates  $x_i$  with  $i = 1, \dots, N$ . For an arbitrary partition  $\lambda = (1^{m_1}, 2^{m_2}, \dots)$  the matrix becomes diagonal after a  $U(1)^{m_1} \times U(2)^{m_2} \times \dots$  rotation with new eigenvalues  $\tilde{x}_i$  for  $i = 1, \dots, l(N)$ . The comparison of this matrix description with the Dunkl operator, identifies the coordinates of (3.17) with the eigenvalues  $\tilde{x}_i$ . It is not difficult to understand the origin of the noncommutative structure of matrix theory. To start with the simplest possible example, decompose the hermitian matrix  $X^\mu$  into Fourier modes of the matrices  $U, V \in U(N)$ , satisfying the relations

$$U^N = V^N = 1 \quad \text{and} \quad UV = qVU, \quad (3.19)$$

with  $q = \exp(2\pi i/N)$  [3]. The two group elements are the generators of  $SU(2)$  embedded into  $U(N)$ , why the Fourier modes depend on two variables, corresponding to the two-dimensional space the root vector lives in. Now the matrix field  $X$  can be written as

$$X = \sum_{n,m} x_{nm} U^n V^m \quad (3.20)$$

or alternatively in the lattice of the root space

$$x(p, q) = \sum_{n,m} x_{nm} e^{\frac{2\pi i}{N}(np+mq)} , \quad (3.21)$$

where the noncommuting coordinates  $(p, q)$  determine the length of the weight lattice. From this, the generalisation to groups of higher rank and different type is obvious and well understood. At least for simply laced algebras, the suggestive form (3.21) has the generalisation

$$f(x) = \sum_{w \in W(R)} f_w q^{<w, x>} , \quad (3.22)$$

with  $W(R)$  the weight space of the roots  $R$  embedded into  $U(N)$ . The noncommutative structure of the matrix theory introduces the new parameter  $q = \exp(2\pi i/N)$  and generalises the Jack  $J(x; \beta)$  to Macdonald polynomials  $J(x; t, q)$ . But in the following we will only adopt the formalism of commutative geometry and set the value of  $q$  to one. To do so, one first has to set  $q = t^\beta$  and then take the limit  $t \rightarrow 1$ . The exact relation thus follows from [22]

$$J_\lambda(x; \beta) = \lim_{t \rightarrow 1} \frac{J_\lambda(x; t, q = t^\beta)}{(1-t)^{|\lambda|}} . \quad (3.23)$$

Apart from the noncommutative structure, matrix theory gives an efficient access to the explicit construction of the higher twist states in the Green-Schwarz formulation [23]. It is a special feature of two-dimensional supersymmetry that the Lagrangian not only consists of single particle multiplets, but also contains twisted multiplets, which play a key role in the calculation of the Bekenstein-Hawking entropy [24,25,26]. The only drawback of the Green-Schwarz formulation is the discrete light cone limit, analogous to the large  $N$  limit of the  $AdS/CFT$  description. But, as stated above, the Liouville action not only gives a valid description of the large string but also of the short string [7], and there is no reason to assume that things will be different in the twisted sector. The primary twisted states are formulated in terms of the  $\mathcal{N} = (2, 2)$  topological conformal field theory and therefore have a simple interpretation as cohomology classes of the orbifold  $X^{[N]}$ . Here

we will not go into the details, but refer to [5,6] for an explicit construction and further analysis. For the special case  $N = 1$  the orbifold simply reduces to the original space  $X$ , but now with the cohomology classes of  $X$  as primary states. The translation of a  $(p, q)$  cohomology form  $\omega^A$  and its corresponding primary state in the  $(a, c)$  ring takes the form [5]

$$\Phi^A = \omega_{ab\dots\bar{a}\bar{b}\dots}^A \psi^a(z) \psi^a(z) \dots \psi^{\bar{a}}(\bar{z}) \psi^{\bar{a}}(\bar{z}) \dots . \quad (3.24)$$

As has been observed by Vafa and Witten in [21], the inherited commutation relations of the even or odd numbers of spinors have a simple representation as spinor or bosonic creation operators  $\alpha_{-1}^A$ , with the familiar commutation relations

$$\{\psi_n^A, \psi_m^B\} = \delta^{AB} \delta_{n+m} \quad \text{and} \quad [\alpha_n^A, \alpha_m^B] = n \delta^{AB} \delta_{n+m} , \quad (3.25)$$

after a convenient rescaling of the operators by  $1/\sqrt{Q_5 Q_1}$ . Here the index of the “space-time” runs up to  $\#H^*(X, \mathbf{Z})$ . It is interesting to note that an analogous relation for the untwisted sector has been found in reference [6], with the spacetime index parametrising the four compact dimensions of  $X$ . Assuming that both algebras are independent from each other, similarly to the two-dimensional field description. But this seems to be somewhat unnatural as the conformal field theories have the conformal charges  $c = 6Q_5 Q_1$  and  $c = 6Q_5$ . So that for  $Q_1 = 1$  both descriptions are allowed, which is in contrast to the result found above, that the large string is part of the untwisted sector and not of the twisted one. A possible solution to this problem is the introduction of a double index for the operators  $\alpha_{N, Q_1}^A$ , which takes care of the operator mode  $N = Q_5 Q_1$  as well as of the number of D1-branes  $Q_1$  with corresponding conformal charge

$$c = c(N, Q_1, \#H^*(X, \mathbf{Z})) , \quad (3.26)$$

which has the property to change by a factor of 6 for  $K3$  (and 4 for  $T^4$ ), if the D1-brane charge exceeds the boundary of  $Q_1 = 1$ .

Up to now, we have only focused on the primary states of the twisted sector, which are the cohomology elements of  $X$  and its tensor products. The introduction of further D1-branes changes this picture as additional singular regions of the orbifold are blown-up and increase the dimension of the moduli space. The blow up modes of the singular CFT are marginal deformations of the twisted sector, whose number for the  $D5/D1$  brane system are easily calculated, because all states of higher twists than  $\mathbf{Z}_2$  are irrelevant. This results in a drastic reduction of the possible Young diagrams, as for a stack of  $N = Q_5 Q_1$

branes only the two partitions  $\lambda_1 = (1^N)$  and  $\lambda_2 = (1^{N-2}, 2^1)$  are related to marginal deformations. Instead of an explicit construction of the vertex operators [6,23], we will use the more efficient description of the Liouville theory in combination with the shift operator (3.17) in Section 5. Following the above example [9], there are 24 cohomology elements of the primary states and two further ones from the orbifold  $K3^{[2]}$ . Knowing that all non-chiral primaries are tensor products of the differential forms, they can be identified with the two partitions  $\lambda_1$  and  $\lambda_2$  and its reduced tensor products. Introducing an additional number  $Q_3$  of D3-branes along the lines of 2.2 reduces the conformal weights of the twist operators by  $1/Q_3$  and a thus larger number of marginal deformations have to be taken into account. The consideration of D3-branes is only one example, where the explicit construction of the vertex operators in terms of the CFT shows up the limits of an analysis along these lines.

#### 4. The Moduli Space

In the previous paragraph we made use of the Virasoro algebra of the free field realisation of the Liouville theory for the twisted and untwisted sector. From a mathematical point of view this construction has been known for a long time [12,13,14] and entered the physical discussion in [21]. In connection with the McKay correspondence it successfully explained, why affine Lie algebras arise after quantising gauge theories. But this is of minor interest here. In this section we will show how the previous  $AdS_3/CFT_2$  discussion can be related to the more general case of intersecting branes in a flat background space. The first part of this section therefore analyses the moduli space of intersecting branes. A special example thereof is the matrix interpretation of M theory with the DBI action as the string theoretic approximation. But the consequences of the quantisation are not considered until the second part. This leads to the Virasoro description of the underlying moduli space and introduces the anomaly term of D-branes. The fusion of the DBI action with the Chern-Simons term as the D-brane charge anomaly will be the final result of Section 5.

#### 4.1. The Coulomb and Higgs Branch in Field Theory

The D-brane action contains basically two superficially different parts, one is the DBI action or kinetic energy of the brane, while the second contribution cancels the anomalous D-brane charge by a Chern-Simons term. As already noted in the introduction, quantising the DBI action is an unsolved problem, but even worse, the expansion of the determinant makes a perturbative analysis impossible. A more appropriate approach was found in [3] using the language of matrix theory. In the last section we already introduced the 1+1 dimensional  $SU(Q_5)$  gauge theory with  $Q_1$  instantons and the  $AdS_3/CFT_2$  correspondence as alternative effective  $D5/D1$  brane description. Whereas the matrix description proved to give an elegant connection between representations of the symmetric group  $S_N$  and the twisted states. For this discussion one must not forget that each representation corresponds to one specific D-brane partition. This way, matrix theory allows a simple mapping between the geometric picture of intersecting flat branes and an effective gauge theory with supermultiplets. Although the quantisation of both formulations is not known, the field theoretic description allows a simple investigation of the underlying moduli space [27,28,17,18] and gives a better understanding of the creation operator  $B_k^+$ . Our main interest in this article is the  $D5/D1$  brane system, which fixes our discussion to the type IIB sector in string theory, but is no principal limitation. In terms of [3], the potential of the matrix action is described by the Lagrangian

$$L = \frac{1}{2\pi\alpha'^2 g} \text{tr} \left( \frac{1}{4} [X^\mu, X^\nu]^2 + \frac{1}{2} \bar{\psi} \gamma_\mu [X^\mu, \psi] \right), \quad (4.1)$$

where the noncommuting matrix fields  $X^\mu$  are representations of  $U(N)$  as in the previous section. The Fourier decomposition (3.20) introduces an infinite number of modes  $x_{nm}$  and consequently infinitely many terms contributing to the kinetic energy of the brane. To recover the classical DBI action, one only has to expand the commutator relations of the matrix fields to first order. The case of interest is the D1-brane for which the expansion of  $N$  separated single branes reduces to

$$\begin{aligned} [X_0, X_1] &= 2\pi i(1 + 2\pi\alpha' F_{01} + \dots), & [X_\mu, X_n] &= 2\pi i D_\mu X_n + \dots \\ [X_\mu, \psi] &= 2\pi i D_\mu \psi + \dots, \end{aligned} \quad (4.2)$$

with the index  $\mu$  running over the bulk coordinates and the spacetime index  $n$  of the D1-brane. Further powers of  $\alpha' F_{01}$  are contained in the higher order terms of the expansion.

The leading term of (4.1) is proportional to the surface density of the brane and reproduces the DBI action

$$S = T_p \int_{\partial X} \text{Tr} \sqrt{\det [G_{nm} + 2\pi\alpha' F_{nm}]} + \dots, \quad (4.3)$$

where we have set the NS B-field to zero. Next, we separate the metric term  $G$  from the determinant and rename the residual curvature  $\mathcal{F} = G^{-1}F$ . The Lagrangian then takes the simple form

$$\text{Tr} \sqrt{1 + \mathcal{F}} \quad (4.4)$$

under the additional assumption  $2\pi\alpha' = 1$ . Written in this simple form the DBI action reveals its topological structure, depending on three contributions.

The derivation of (4.4) started from the assumption of  $N$  separated branes with a  $U(1)$  gauge freedom on each surface. We are not able to show from matrix theory alone that an analogous formula holds in the case of higher gauge groups, but we will give an indirect argument at the end of this subsection that in principle the structure is still correct. It is possible to generalise  $\mathcal{F}$  formally to higher gauge groups. The first thing we learn from (4.4) is that the discriminant is the Chern class  $c(\mathcal{F})$  of a gauge bundle with structure group ranging from  $U(1)$  to  $U(N)$ . Consequently, the square root expresses the Chern class of a single D-brane. Probing the stack of  $N$  separated branes by the interaction with an external brane follows now from the relation  $c(\mathcal{F} + \mathcal{G}) = c(\mathcal{F})c(\mathcal{G})$ . A special case is  $\mathcal{F} = \mathcal{G}$  where the square root of (4.4) reduces to the simple Chern class  $c(\mathcal{F})$ . In principle, this form also allows interactions between branes of the stack itself, which leads us to the trace over the Chan-Paton factors. Branes interacting among themselves increases the rank of the gauge group and thus the rank of the underlying vector bundle, but it also reduces the number of possible combinations of the probing brane with the stack. Taking into account that we have set the B-field to zero and the branes are free to move relatively to each other. So the action (4.4) is only the first term in a series of different partitions, where the first part corresponds to the Coulomb branch  $\lambda = (1^N)$  of the last section with one D1-brane interacting with a stack of  $N$  D5-branes. Now, the complete action along the lines of (4.4) corresponds to the Chern class of all partitions of branes, which can formally be written as

$$S = \int_{\partial X^{[N]}} \sqrt{c(u)} \quad \text{with} \quad u \in K(X, \mathbf{Z}) \quad (4.5)$$

and  $c(u) \in \text{End}(H^*(X^{[N]}))$  as Chern class of the Hilbert scheme at order  $N$ . As the D-branes are wrapped around the Hilbert scheme  $X^{[N]}$ , the integral has to be evaluated over

its boundary  $\partial X^{[N]}$ . In the next section we will demonstrate, how the integral over the “boundary of the Hilbert scheme” is to be calculated recursively in the framework of the Virasoro algebra. Unfortunately, we have no representation in terms of gauge fields similar to the D-brane anomaly, but knowing the exact kinetic energy is always the first step of perturbation theory, we will use the propagator  $c^{-1}(u)$  to calculate the residual brane dynamics in Section 5. But neither the matrix description nor the *AdS/CFT* correspondence alone provides an applicable method to do this.

The DBI action restricts the analysis of the moduli space to the Coulomb sector and leaves the more general Higgs branch and the intermediate states completely untouched. Furthermore we still have to show that the assumption of (4.5) is correct. But it is not very reasonable to try and generalise the DBI action. Instead we will combine the ADHM description of intersecting D-branes in matrix theory [28] with the Virasoro algebra [12,14]. This way one avoids the expansion of matrix fields in terms of gauge interactions (4.2) and thus the problem of analysing the DBI action, which is correct only in case of the Coulomb branch of the underlying field theory. Basically the interpretation of intersecting matrices and their moduli spaces has been done in [28,29] for the commutative as well as the noncommutative case. Therefore we will not repeat the actual correspondence and instead start with the relevant mathematical formulation of moduli spaces as hyperkähler moment maps. As a four-dimensional example consider the space  $\mathbb{C}^2$  with the Hilbert scheme  $(\mathbb{C}^2)^{[N]}$  as the moduli space of zero-dimensional subschemes [13]. Because *K3* has a local representation as an ALE space, this example can be seen as an approximation to the above problem. Following the ADHM construction of the moduli space of charge  $Q_1$  instantons with  $SU(Q_5)$  symmetry, one introduces two matrices  $B_1, B_2 \in \text{Hom}(V, V)$  and two vectors  $I \in \text{Hom}(W, V)$  and  $J \in \text{Hom}(V, W)$  in the complex hermitian vector spaces  $W = \mathbb{C}^{Q_5}$ ,  $V = \mathbb{C}^{Q_1}$ . Now, the actual moduli space  $\mathcal{M}(Q_5, Q_1)$  is determined by the set  $(B_1, B_2, I, J)$

$$\begin{aligned}\mu_{\mathbf{R}} &= \frac{i}{\sqrt{2}} ([B_1, B_1^+] + [B_2, B_2^+] + II^+ + J^+J) \\ \mu_{\mathbf{C}} &= [B_1, B_2] + IJ ,\end{aligned}\tag{4.6}$$

where the space  $\mathcal{M}$  is defined as the  $U(V, \mathbb{C})$  invariant space of

$$\mathcal{M}(Q_5, Q_1) = (\mu_{\mathbf{R}}^{-1}(0) \cap \mu_{\mathbf{C}}^{-1}(0)) / U(V)\tag{4.7}$$

with the group  $U(V)$  acting on  $(B_1, B_2, I, J)$  by the relation  $g \cdot (B_1, B_2, I, J) = (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1})$ . For a further discussion of these equations we refer to

[13,28,29] and references therein. The analysis of the moduli space with respect to the vector  $J$  brings us to the consideration of noncommutative geometry, which is not of primary interest in the discussion here. We will set  $J = 0$  for the time being and postpone the discussion to a later section. What is important at the moment is the connection between the matrix description of M theory and the moduli space determined by the ADHM description. As we are not interested in special solutions of (4.6), it is sufficient to classify the possible solutions without actually calculating their matrices [12]. In a first step we decompose the holomorphic vector space  $V$  into its weight spaces by the torus action  $\lambda : T^2 \rightarrow U(V)$ , satisfying the conditions

$$\begin{aligned} t_1 B_1 &= \lambda(t)^{-1} B_1 \lambda(t) \\ t_2 B_2 &= \lambda(t)^{-1} B_2 \lambda(t) \\ I &= \lambda(t)^{-1} I, \end{aligned} \tag{4.8}$$

with the two coordinates  $(t_1, t_2) \in T^2$  of the torus. By decomposing the matrices into  $B_1, B_2$ , the vector space  $V$  separates into the weight spaces  $V = \sum_{k,l} V(k, l)$ , defined by

$$V(k, l) = \{v \in V | \lambda(t) \cdot v = t_1^k t_2^l v\}. \tag{4.9}$$

In this weight space the defining relations (4.6) of the moduli space take the simple form

$$\begin{aligned} B_1 : V(k, l) &\rightarrow V(k-1, l) \\ B_2 : V(k, l) &\rightarrow V(k, l-1) \\ I : W &\rightarrow V(0, 0). \end{aligned} \tag{4.10}$$

Since we set  $J = 0$ , the matrices satisfy the commutation relation  $[B_1, B_2] = 0$  as a consequence of the second equation of (4.6). In combination with the finiteness of the vector space  $V$ , the operation of the matrices cuts a finite grid out of the two-dimensional net of the weight spaces  $V(k, l)$ , which in [12,14] has been connected to Young diagrams, the graphical representation of partitions  $\lambda = [\lambda_1, \lambda_2, \dots]$  as introduced in Section 3. Actually, the decomposition of  $V$  into weight spaces is the abstract formulation of the matrix expansion (4.2) with the square root of the Chern class of  $V$  as the generalisation of the DBI action. The Coulomb branch for example decomposes into  $0 \rightarrow V(0, 0) \rightarrow V(0, 1) \rightarrow \dots \rightarrow V(0, N) \rightarrow 0$  with each space of dimension one and Chern class  $\det(1 + \mathcal{F})$ .

The discussion of matrix theory and its moduli space joins the results of the  $AdS_3/CFT_2$  discussion of the last section, where we have shown that the null vectors



of the Liouville theory describes on the one hand the partitions of branes and on the other hand the cohomology ring of the Hilbert scheme in terms of a Virasoro algebra. Therefore we argue, that the combination of both theories allows the description of M theory not only in the large  $N$  limit or at the boundary of the  $AdS$  space but for all values of  $N$ . Here we have to mention, that the discussion of K-theory for D-brane / anti D-brane interactions suggests a small limitation to this assumption, as the statement is only correct for  $N \geq 10$ . We believe that this restriction has the same origin as the  $Q_1$  independence of the conformal charge, a puzzle we cannot solve so far.

#### 4.2. Hilbert Schemes and the Virasoro Algebra

In the previous section we have shown that the moduli space of the multiplets as determined by matrix theory are intimately connected with the Virasoro algebra representation of Young diagrams. In this section we will tie some of the loose ends together, as the connection between the degree of a partition and the anomaly condition of D-branes or the signature of spacetime for the Virasoro algebra and the moduli space. The results entering our discussion are based on [13,30] and we will repeat some of the ideas in the context of our analysis of D-brane interactions.

For our generalisation of the Green-Schwarz description of the  $D5/D1$  stack we claimed that the spacetime dimension of the Virasoro algebra of the twisted sector is determined by the number of cohomology elements of the manifold wrapped by the D5-brane (3.24). But there is a slight difference compared to the mathematical point of view, where the dimension is fixed by the Neron-Sevéri lattice. As an explicit example consider the case  $X = K3$ . The dimension of this lattice is basically the number of  $H^{1,1}(X, \mathbf{Z})$  elements, which is 20 for  $X = K3$ , but the whole number of cohomology elements is 24. Surprisingly, string theory combines the information about the Neron-Sevéri group with the number of D-branes and the gauge group of the string theory into the D-brane anomaly [30]. For the example of  $K3$  this can be seen as follows. The intersection form of  $H^2(K3, \mathbf{Z})$  is isometric to  $((-E^8) \oplus H) \times ((-E^8) \oplus H)$  and defines the gauge group of the effective (bosonic) string theory in  $24 + 2$  dimensions. As long as the considered manifold has a complex structure, the Hodge decomposition of the  $H^2(\mathbf{Z})$  forms is related to the decomposition under the complex structure, since the antiselfdual cohomology elements are related to  $H^{1,1}(\mathbf{Z})$ . At least for a stable moduli space, the lattice of selfdual RR fields [1] is now sufficient to determine the Neron-Sevéri lattice. But the WZW term of the D-brane anomaly introduces the number of intersecting D-branes as further information.

Although the example of  $K3$  is interesting, it is not very convincing for our problem of type IIB string theory, as we claimed that the twisted sector and the untwisted one belong to the same algebraic construction and this forces a ten-dimensional space  $(1, 9)$  with conformal charge  $12 = 8 + 4$  and not the  $24 + 2$  dimensions of  $K3$ . We already mentioned the problem with the D1-brane charge before. This puzzle can be reformulated in terms of the moduli space of  $X$  and we believe to have found one possible solution by Enriques surfaces  $Y$ . The universal covering of  $Y$  is  $K3$  with intersection form  $(-E^8) \oplus H$ . Even the cohomology groups  $H^2(Y, \mathbf{Z}) = \mathbf{Z}^{10} \oplus \mathbf{Z}_2$ ,  $h^0 = h^4 = 1$  determine the underlying string theory to be ten-dimensional. Although it is very speculative, we think that the number of D1-branes determines the number of  $Y$  coverings. As the fundamental group of  $Y$  is  $\mathbf{Z}_2$ , this reproduces the ten-dimensional string theory for  $Q_1 = 1$  and at  $Q_1 = 2$  the manifold is basically  $K3$ . A further hint comes from the W algebra analysis itself, as the canonical bundle of  $Y$  is of torsion class  $K_Y^2 = \mathcal{O}_Y$ . It is important to note that D-branes cannot wrap Enriques surfaces, but we think of it as a generating space for the effective string theories.

Before we turn to the discussion of the Virasoro algebra, some further comments on the Neron-Severi lattice  $\text{NS}(X)$  are in order as they uncover some new aspects of the D-brane anomaly and the quantisation of selfdual RR fields. As assumed before,  $X$  is a complex manifold with structural sheaf  $\mathcal{O}_X$ . It follows from the exponential map  $\exp : \mathcal{O}_X \rightarrow \mathcal{O}_X^*$  that the sequence

$$0 \rightarrow \mathbf{Z} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X^* \rightarrow 0, \quad (4.11)$$

gives rise to the cohomology sequence

$$\rightarrow H^1(X, \mathbf{Z}) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbf{Z}) \rightarrow H^2(X, \mathcal{O}_X) \rightarrow, \quad (4.12)$$

where the derivative  $d : H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbf{Z})$  defines the Picard group of  $X$ . If we define  $\text{Pic}_0(X) = \text{Ker}(d)$ , the Néron-Severi group is isomorphic to the quotient

$$\begin{aligned} \text{NS}(X) &= \text{Pic}(X)/\text{Pic}_0(X) \\ &= \text{NS}(X)/\text{Tors}(\text{NS}(X)) \times \text{Tors}(\text{NS}(X)) \\ &= \mathcal{NS}(X) \times \text{Tors}(\text{NS}(X)), \end{aligned} \quad (4.13)$$

where we have separated the torsion part of  $\text{NS}(X)$ . For the following it is important to get a better understanding of the significance of the Néron-Severi group. Therefore we will first give a simple example. An appropriate starting point is the geometric engineering

of string theories. Consider for example the ALE space  $X = \mathbb{C}^2/\Gamma$  with nondegenerate lattice  $\Gamma = v_1\mathbf{Z} + v_2\mathbf{Z} + v_3\mathbf{Z} + v_4\mathbf{Z}$  cutting out the curve  $C = \sum a^{ij}S_{ij}$  generated by the cycles  $S_{ij}$ . Choosing a differential form  $\omega$  on  $X$ , we get for each of the four cycles

$$\int_{S_{ij}} \eta = \alpha_i \beta_j - \alpha_j \beta_i . \quad (4.14)$$

Thus for the curve  $C \subset X$  we get

$$\int_C \eta = \sum_{i < j} a^{ij} (\alpha_i \beta_j - \alpha_j \beta_i) = 0 . \quad (4.15)$$

The solutions of this relation determines the Picard lattice. Here the answer is quite trivial as  $\Gamma$  is nondegenerate and thus  $\text{Pic}(X) = \text{Pic}_0(X)$ . But in general there are nontrivial solutions generating the  $\text{NS}(X)$  lattice. The skew symmetric product entering the above calculation is known in field theory as the Dirac quantisation condition and relates two elements in the homology. The subgroup  $\text{Pic}_0(X)$  determines exactly those elements for which no duality relation exists. But if the Néron-Severi group is non trivial some of the numbers  $(\alpha_i, \beta_i)$  have to vanish. The result (4.15) takes a rather suggestive form if one identifies the numbers  $(\alpha_i, \beta_i)$  with its cohomology elements in  $H^2(X, \mathbf{Z})$  and the differential form  $\eta$  with the  $\theta$ -term in gauge theory  $F * F$ . In string theory this intersection form is known as the anomaly condition

$$\langle x, \bar{y} \rangle = \int_X G(x) \wedge *G(\bar{y}) = \int_X G(x) \wedge G(y) , \quad (4.16)$$

but as shown in [30] K-theory always combines the anomaly term with the kinetic energy of the RR fields

$$\Theta(x, y) = \exp\{(x, y)\} = \exp\{\langle x, y \rangle + \tau \langle x, \bar{y} \rangle\} , \quad (4.17)$$

into a theta function  $\Theta(x, y)$  for  $x, y \in K(X)$ , where we understand the gravitational anomaly as included. This theta function describes more than just the Picard lattice of  $X$ . Actually, it combines the degree of D-brane partitions, of which we studied the example  $N = Q_5 Q_1$  with a generalisation of the NS lattice, as the gravitational anomaly introduces a further  $\mathbf{Z}_2$  dependence. The knowledge of the theta function (4.17) already determines the topological information we will need for the construction in the next section. But before we turn to the algebraic aspects of the moduli space, let us emphasize at least two important aspects of (4.17).

The main motivation for the introduction of the NS lattice  $\Gamma$  was its appearance as the metric of the spacetime in the Virasoro algebra. Now the action of S-duality maps the lattice to its inverse  $\Gamma^*$  and T-duality, as it interchanges the RR fields in (4.17), maps it to its transpose  $\Gamma^{\text{Tr}}$ . Furthermore, the intersection (4.16) at level zero,  $N = 0$ , determines the conformal charge of the Virasoro algebra and since the *AdS/CFT* correspondence is correct only in the large  $N$  limit, the only information we get from the field theoretic description of M theory is that of an  $N$ -fold covering of the manifold  $X$  but not of  $X$  itself. This again underlines the necessity to understand M theory for finite  $N$ , because any manifold  $X$ , whose infinite covering is isomorphic to  $K3$ , gives an equivalent description in the field theory formulation. The class of Enriques surfaces we mentioned above is one example.

The spacetime dimension of the Virasoro algebra is actually only the first step in the analysis of the moduli space. To extract information from this construction one still has to find a way as to combine the modes in such a way to relate the topological information to the algebra. This is simple enough from the point of string theory, as the Virasoro algebra is only an intermediate step in the process of quantisation. The elliptic genus and the partition function of the torus  $T^2$  are only two examples of how to extract the topological information from the algebra. The by far more general objects are the vertex operators of fields, which we now turn to. Here again, it is interesting to note how similar the mathematical description is when compared to the physical one. The Virasoro modes of the physical fields have to respect additional conditions originating from compactifications or boundaries, whereas the moduli space determines a kind of effective Virasoro algebra, which again defines a “field”. How to calculate this effective algebra is the subject of the next section. Here we will only give an idea of the mathematical construction behind the vertex operator. Its formal definition is simply the infinite sum [13]

$$\begin{aligned} V_{\text{NS}(X)} &= \sum_{c_1 \in \text{NS}(X), ch_2} H_{\text{NS}(X)}(\mathcal{M}(c_1, ch_2)) \\ &= \sum_n H_{\text{NS}(X)}(X^{[n]}) \otimes \mathbf{Q}[\text{NS}(X)] , \end{aligned} \tag{4.18}$$

where the order of the Hilbert scheme  $X^{[n]}$  is determined by

$$n = -\text{ch}_2(X) + \frac{1}{2} \langle c_1, c_1 \rangle_X . \tag{4.19}$$

As shown in [31] this is the product of the Mukai vector with the canonical sheaf  $\mathcal{O}_X$  or simply the bundle of wrapped D5-branes, intersecting with the line bundle of D1-branes. Written in terms of the D-brane anomaly (4.16) it takes the form

$$N = \frac{\chi(X)}{24} - \frac{1}{2} \int_X G \wedge G, \quad (4.20)$$

which is related to (4.19) by  $n = -24 N$ . One further interpretation of the anomaly number  $N$  originates from the K-theoretic picture. An alternative way to write the D-brane anomaly (4.16) is by insertion of the equation of motion for the RR fields with the result

$$\langle x, \bar{y} \rangle = \int_X \hat{A}(X) \text{ch}(x) \text{ch}(y), \quad (4.21)$$

where the elements  $x \in K(X)$  are represented by vector bundles. Take for simplicity  $x \in [E]$ , the class of D5-brane bundles and  $y \in L$ , the class of D1-brane line bundles. The degree of the D-brane partition  $N$  is then determined by the monodromy around the effective divisor  $D$  of the line bundle on  $X$  defined by a shift  $[E] \rightarrow [E] \otimes \mathcal{O}_X(D)$ . As long as the D-branes satisfy the BPS bond, both formulas (4.16) and (4.21) give an integer anomaly number  $N$ . As is well known, this is no longer true if a selfdual NS B-field is turned on, corresponding to a shift in the field tensor  $F \rightarrow F + B$ . Moving around the divisor takes the anomaly number  $N$  into non integer values and thus contradicts our assumption that  $N$  is the degree of a partition. But we believe that this problem has a simple solution. As for  $B^+ \neq 0$ , the system of interacting D-branes are no longer BPS, the winding numbers around  $X$  and the singularity of the D1-brane  $D$  need not to be integer any more. From (4.16) we know that the RR fields generate a two-dimensional torus similar to the Picard torus. Let  $[C_5]$  and  $[C_1]$  be the two cycles of this torus with a foliation  $[C] = Q_5[C_5] + Q_1[C_1]$  in the case of BPS branes. Turning on a B-field, the cycles do not close, up to the noncommutativity parameter  $\theta$ . For the same charges the homology element  $[C]$  now becomes  $[C] = Q_5[C_5] + Q_1[C_1] + \theta Q_5 Q_1 [C_5 \cap C_1]$ . It is therefore necessary to move back along the path by an amount of  $\theta N [C_5 \cap C_1]$ . But moving in the opposite direction is equivalent to the monodromy around the inverse line bundle  $L^{-1}(\theta)$  represented by an anti D1-brane of charge  $\theta$ . In the next section we will make use of this construction. But before that, we have to explain how moduli parameters such as  $\theta$  are to be incorporated into the algebra.

This brings us back to the effective Virasoro algebra of the moduli space. One example already entered the discussion in the analysis of the partition of the  $D5/D1$  stack (3.12).

After the by now common sign change in the Virasoro modes, the commutation relations take the form [12]

$$[q_n^i, q_m^j] = (-1)^n n \delta^{ij} \delta_{n+m} . \quad (4.22)$$

The calculation of this effective algebra was simple, because of the analogy between the allowed partitions of the stack of D-branes and the generating function of the elementary symmetric functions (3.11). As we will see in the next section the similarity between (3.9) and the Chern classes of the moduli space is no coincidence. Actually, the connections between string theory and the algebra of vertex operators are far more numerous and we refer to [13] and references therein.

## 5. W Algebras and Hilbert Schemes

In this chapter we will finally construct the shift operator analogous to (3.17) in creation modes of the free field representation of the Liouville theory. Following the discussion of Section 3, the bosonic part contains the basic structure, so that as a first attempt we can restrict the discussion to the case of  $K3$ . We will make two important assumptions here, first the noncommutative part of (3.17) will be omitted and furthermore the spacial dimensions of the operation modes and thus of the operators itself are not indicated in the formulas. The last assumption is made because the correct structure of the conformal charge is not known to us for general D1-brane charges and thus has to be omitted. But the formulas will be true for the untwisted as well as the twisted sector separately so that the results make sense in both cases. With this understood we begin with the analysis of the shift operator and its generated W algebra. It is then a simple task to generalise this result to the more general supersymmetric case of a  $W_\infty(\lambda)$  algebra, which finally includes the complete cohomological structure of the twisted sector even for manifolds different from  $K3$ . As a test that all states of Section 3 are reproduced correctly, we will reconsider the simple case of a stack of  $D5/D1$  branes and calculate the vertex function and the corresponding commutation relation of the resulting operator modes (4.22). As an outlook of the advantage of this formulation we consider the three point amplitude for one D3-brane inserted into the stack of  $D5/D1$  branes, a result which has been calculated by M. Lehn [15]. Finally we speculate on a correspondence between six-dimensional Calabi-Yau manifolds and  $W_3$  algebras.

### 5.1. The $W_\infty(\lambda)$ Algebra

In Section 3 we introduced Jack polynomials  $J_\lambda(x; \beta)$  to compare the null vectors of minimal models and its descendants with representations of the symmetric group  $S_N$ . These specific functions are the solutions of the Calogero-Moser model and have the Dunkl operator (3.17) as a kind of covariant derivative. As the analysis of [22] shows, Jack polynomials can be constructed by creation operators  $B_k^+$  from the Dunkl operator as symmetry generating operator that defines the corresponding Hilbert space. How are the Hilbert spaces of the minimal model and the integrable system related, and what is the connection between the coordinates  $x_i$ , the string modes  $a_n$  of (3.2) and the eigenvalues of the matrices (3.19) in matrix theory?

In a first step, to identify the relation between the coordinates with the matrices (3.21) we look at the second part of (3.17). It is suggestive to compare the matrix elements of  $x(p, q)$  with the coordinates  $x_i$  of the Dunkl operator. In the Higgs sector the matrix field  $X$  can be chosen to be diagonal so that the identification relates the  $x_i$  with the eigenvalues of the matrix. But one has to take care for the possible breakdown of the  $U(N)$  symmetry. The matrix  $K_{ij}$  exchanges the modes  $(n, m)$  in (3.21) and commutes the spacial coordinates  $(p, q)$ . To compensate the noncommuting part, one has to introduce two new indices for the coordinates  $(p, q)$  and an additional parameter, which leads to the Macdonald polynomials, as already mentioned in Section 3. For this reason we will assume that the matrix fields only depend on one spacial coordinate  $\theta$ , parametrising the sphere  $S^1$ . With these simplifications the creation operator  $B_k^+$  takes the simple form of the Liouville action (3.2). To see this, identify the coordinates in (3.17) with the eigenfunction of  $S^1$ . As stated in Section 3, the operator  $B_k^+$  contains basically terms of the form  $(D_i + \omega)x_i$ . The first part then reduces to the derivative  $\partial/\partial\theta$ , whereas the sum vanishes because of the cyclic structure of  $x_n$ . But now the affine Lie algebra of  $S^1$  is the Virasoro algebra, and the shift operator, with its origin in matrix theory, can be reformulated in creation modes  $a_n$  of the minimal model (3.2).

To get an operator, independent of further indices, we make use of the combination  $\sum_n a_{-n} \mathcal{L}_n$  and get the final form

$$\mathcal{D} = \frac{1}{2} \sum_n a_{-n} L_n - \frac{1}{2} \alpha_0 \sum_n (n+1) a_{-n} a_n . \quad (5.1)$$

The advantage of this representation, instead of the more elementary form of  $\mathcal{L}_n$ , is its symmetry in the index  $n$ . In the case of commuting variables, it is easy to give the general

action on the string modes  $a_n$ . The operator  $\mathcal{D}$  acts as a derivative, why we will call  $\mathcal{D}^k a_n$  the  $k$ -th derivative and denote it by  $a_n^{(k)}$ . For small  $k$  we will also make use of the notation  $a'_n = \mathcal{D}a_n$ . The first derivative and its commutator take the form

$$\begin{aligned} a'_n &= -nL_n + \alpha_0 n(n+1)a_n \\ [a'_n, a_m] &= nm(a_{n+m} - \alpha_0(n+1)\delta_{n+m}) . \end{aligned} \quad (5.2)$$

The commutation relation generates a new algebra with many interesting features. For example, in the case that the Liouville theory degenerates to an ordinary conformal field theory at  $\beta = 1$ , the derivative reduces to the operator  $-nL_n$ , and all possible states, generated by  $\mathcal{D}$  degenerate to the classical vertex operator of string theory. Another interesting feature is the degeneracy at  $n = -1$ . In Section 3 we observed a contradiction for the case  $Q_1 = 1$ , where classically the untwisted as well as the twisted conformal description are valid conformal field theory descriptions of the moduli space. We see, that for any partition of  $N = Q_5 Q_1$  the critical case of  $n = -1$  is independent of the Liouville part of the action. Therefore the operator mode  $a'_{-1}$  can only shift the index by one, but does not change the actual ground state. Although this does not solve the problem of the conformal charge and its independence concerning the D1-brane charge, it circumvents the only critical point in our discussion by eliminating this dependence for  $a_{-1}$ . The other characteristics of the derivative  $\mathcal{D}$  are less obvious. In anticipation of a discussion in the next section, we will show here that the relations (5.2) and thus the operator (5.1) are all we need to calculate the Jack polynomials.

In the previous sections we argued that the stack of  $D5/D1$  branes only allows  $\mathbf{Z}_2$  twists for any number of  $N > 2$ . If our arguments are correct, this twist of the fields corresponds to the first derivative of the string modes, which already determines the action of the operators  $a_n^{(k)}$  on the vacuum to be zero, because the ground state cannot be twisted. The ansatz for the other state vectors, which will be motivated in the next section, is the polynomial

$$J_{(1^n)}(a, \beta = 2) = \frac{1}{n!} (a_{-1} + a'_{-1})^n |0\rangle . \quad (5.3)$$

The single first derivative  $a'_{-1}$  interweaves the D1-brane with the remaining stack of D5-branes. Calculating the first three terms gives the result

$$\begin{aligned} J_{(1^1)} &= a_{-1} \\ J_{(1^2)} &= \frac{1}{2} (a_{-2} + a_{-1}^2) \\ J_{(1^3)} &= \frac{1}{3} \left( a_{-3} + \frac{3}{2} a_{-2} a_{-1} + \frac{1}{2} a_{-1}^3 \right) , \end{aligned} \quad (5.4)$$



which reproduces the primary states (3.5) up to a sign reversal of the string modes. As expected, all these states belong to the Higgs sector and after a convenient rescaling can be rewritten in the simple form of Schur polynomials. The generating functions (3.11) now allows an even more suggestive form as the vertex operator

$$e^{-\Phi(z)} = \exp \left( - \sum_{n \in \mathbf{Z}} \frac{(-1)^n}{n} q_n z^n \right), \quad (5.5)$$

with the new bosonic operator modes  $q_n$ , obeying the commutation relation (4.22). This vertex operator can be understood as the effective field of the  $D5/D1$  brane system with the partition of D1-branes along the stack of D5-branes, similar to an effective action in field theory. In the next section we will be concerned with a better understanding of this construction in the framework of moduli spaces [13,15].

For the above example the guess of a generating function has been quite simple, but for more general cases one needs a strategy to solve the equation (5.3). In a first step one has to reformulate the defining equation in the Virasoro modes as a recurrence relation in a polynomial ring. With the substitution of  $a_{-1}$  by the variable  $x$ , equation (5.3) becomes

$$A_n(x) = (x + xD)A_{n-1}(x), \quad A_n(x) = \sum_{k=0}^{\infty} g[n, k] x^k \quad (5.6)$$

with the action of  $D$  determined by (5.2). Inserting the ansatz for  $A_n(x)$  one finally obtains a recurrence relation in the coefficients of  $g[n, k]$

$$g[n, k] = g[n-1, k-1] + k g[n-1, k-1]. \quad (5.7)$$

Which again is easily solved in the index  $n$  by the generating function  $B_k(y) = \sum g[n, k] y^n$  and thus determines the coefficients  $g[n, k]$ . It is interesting to be a bit more general and to modify the recurrence relation (5.7) by an arbitrary integer  $\alpha - 1$

$$g[n, k] = (\alpha - 1) g[n-1, k-1] + k g[n-1, k-1] \quad (5.8)$$

to obtain the equation

$$\begin{aligned} \sum_n g[n, k] y^n &= (\alpha - 1) y \sum_n g[n-1, k-1] y^{n-1} + k y \sum_n g[n-1, k-1] y^{n-1} \\ B_k(y) &= y(\alpha + k - 1) B_{k-1}(y) \\ &= y^k (\alpha + k - 1) \dots (\alpha - 1). \end{aligned} \quad (5.9)$$

The only nonvanishing coefficient  $g[k+1, k]$ , inserted into the generating function gives the result

$$\begin{aligned}
V(z) &= \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k \\
&= \sum_{k=0}^{\infty} \frac{(k+\alpha-1)! y^k}{(\alpha-1)! k!} z^k \\
&= \frac{1}{(1-yz)^\alpha} \\
&= \exp\left(-\alpha \sum_{n=0}^{\infty} \frac{y^n}{n} z^n\right).
\end{aligned} \tag{5.10}$$

This derivation, being valid for arbitrary values of  $\alpha$ , it allows in a convenient way to calculate the generating function in dependency of the parameter  $\beta$ .

All the primary states of the Higgs branch can be reproduced in the form of (5.3), but the construction of the nonprimary states is still an outstanding problem. Following the same argument as above, that the  $D5/D1$  brane system demands first order derivatives only, the missing states are determined by

$$\begin{aligned}
J_{(1^3)} &= \frac{1}{3} a_{-3} + \frac{1}{2} a_{-2} a_{-1} + \frac{1}{6} a_{-1}^3 \\
\frac{1}{1!} a'_{-1} J_{(1^2)} &= a_{-3} + a_{-2} a_{-1} \\
\frac{1}{2!} a'_{-2} J_{(1^1)} &= a_{-3},
\end{aligned} \tag{5.11}$$

which is in exact agreement with the orbit sum (3.13) after the reversal of signs as above. This shows that all single particle as well as multiparticle states have a representation as an polynomial in derivatives of  $a_n$ .

The restriction of our consideration to D1 and D5-branes made the introduction of the first derivative necessary. But as has already been noted in Section 3, the incorporation of D3-branes along the lines of Section 2 force the introduction of even higher twists and thus higher derivatives in the string modes. It is interesting to look at the general structure, which the derivative  $\mathcal{D}$  generates by its iterative action on the Liouville field  $\phi$ . The first three terms in this sequence, as calculated from the differential operator (5.1) and the explicit form of the Liouville action (2.13) are given by

$$\begin{aligned}
V^{-1} &= \partial\phi \\
V^0 &= \frac{1}{2}(\partial\phi)^2 - \alpha_0 \partial^2\phi \\
V^1 &= \frac{1}{3}(\partial\phi)^3 - \alpha_0 \partial\phi \partial^2\phi + \frac{1}{3} \alpha_0^2 \partial^3\phi.
\end{aligned} \tag{5.12}$$

So each operation by  $\mathcal{D}$  increases the order of the partial derivative  $\partial\phi$  by one. A generator  $V^k$  of this sequence is therefore a bosonic current of charge  $s = k + 2$ . This is the main solution to the original problem, namely to explain the origin of a CFT with conformal charge  $c = 6N$  for any integer  $N = Q_5 Q_1$ , although the original theory only contained a massless field of at most spin two. As suggested by Vafa [8] the elements of (5.12) resemble the lowest currents of a  $W_{1+\infty}$  algebra, a symmetry valid not only for the twisted but also for the untwisted sector, as learned from [6]. W algebras have many interesting features of which we will review only some in the following. As an introduction and for further references we recommend [32,33].

The general *AdS/CFT* correspondence [2] supposes the duality to be exact only in the large  $N$  limit, what excludes all finite dimensional W algebras. Furthermore, we will always need at least one spin-one current, which gives a further restriction on  $W_{1+\infty}$  algebras and its tensor products with at most  $W_\infty$ . The bosonic realisation of the  $W_{1+\infty}$  algebra is sufficient for the description of the moduli space of  $K3$ , but the introduction of  $T^4$  already demands the incorporation of the  $N = 2$  supersymmetric W algebra [32], whose bosonic sector is  $W_{1+\infty} \times W_{1+\infty}$ . The free field Lagrangian of the underlying model is  $L = \bar{\partial}\bar{\phi}\partial\phi + \bar{\psi}\bar{\partial}\psi$  and respects the necessary additivity of the number of cohomology classes and conformal charges. As learned from [33] the analysis of the  $W_{1+\infty}$  algebra is best studied in the fermionic realisation

$$L = \frac{1}{2}\partial\bar{\psi}\psi - \frac{1}{2}\bar{\psi}\partial\psi \quad (5.13)$$

with central charge  $c = 1$ . The resulting currents  $V^k(z)$  form an irreducible basis of the algebra regarding the spin, instead of the multiparticle analysis from *AdS/CFT*. Of course this basis is not unique, but a convenient choice is one for which the currents are quasiprimary states with respect to the energy-momentum tensor. This allows to identify the W algebra currents with the single particle states of the KK modes in a 1-parameter dependent basis

$$V^i(z) = \sum_{j=0}^{i+1} \alpha_j(i, \lambda) \partial^j \bar{\psi} \partial^{i+1-j} \psi, \quad (5.14)$$

with the coefficients

$$\alpha_j(i; \lambda) = \binom{i+1}{j} \frac{(i+2\lambda+2-j)_j (2\lambda-i-1)_{i+1-j}}{(i+2)_{i+1}}, \quad (5.15)$$

where the bracket  $(a)_k$  is the ascending Pochhammer symbol for integer  $a$  defined by  $(a)_k = (a+k-1)!/(a-1)!$ . To compare this representation of the currents with the higher derivatives of the bosonic modes of  $\phi$ , one has to rebosonise the complex fermionic field  $\psi$  the free scalar field

$$\psi = e^\phi \quad \text{and} \quad \bar{\psi} = e^{-\phi} \quad (5.16)$$

The bilinear terms  $\partial^j \bar{\psi} \partial^i \psi$  in the sum (5.14) now take the form

$$\partial^j \bar{\psi} \partial^i \psi = \sum_{k=i+1}^{i+j+1} \frac{1}{k} (-1)^{k-i-1} \binom{j}{k-i-1} \partial^{i+j-k+1} P^{(k)}(z) , \quad (5.17)$$

where the polynomials  $P^{(k)}(z)$  are of Hermitian type:

$$P^{(k)}(z) = e^{-\phi(z)} \partial^k e^{\phi(z)} . \quad (5.18)$$

The expansion of (5.14) with these redefinitions finally gives the currents (5.12) with the Liouville field as free bosonic particle [33] and parameter  $\lambda = -\alpha_0$  as improvement charge. The analog description for  $T^4$  with 8 even and 8 odd cohomology elements makes the introduction of fermionic currents  $G^i(z)$  and  $\bar{G}^i(z)$  necessary [32], and extends the algebra to the  $\mathcal{N} = 1$  supersymmetry  $W_{1+\infty}(\lambda)$ . The  $\lambda$  dependence makes the discussion rather complicated, why we take the parameter to be zero. Then the additional currents take the form

$$\begin{aligned} G^i(z) &= \sum_{k=0}^i \gamma_k(i) \partial^{i-k+1} \bar{\phi} \partial^k \psi \\ \bar{G}^i(z) &= \sum_{k=0}^i \gamma_k(i) \partial^{i-k+1} \phi \partial^k \bar{\psi} , \end{aligned} \quad (5.19)$$

where the expansion coefficients  $\gamma_k(i)$  are given by

$$\gamma_k(i) = \frac{(-1)^k}{2^i} \frac{(i+1)!}{(2i+1)!!} \binom{i}{k} \binom{i+1}{k} . \quad (5.20)$$

Actually it is not the field representation of the W algebra we will need, but the algebra of modes and the mode depending conformal charge. The final result for the  $W_{1+\infty}$  algebra with  $a$  and  $\lambda$  set to zero is then [33]

$$[V_m^i, V_n^j] = \sum_{k \geq 0} q^{2k} \tilde{g}_{2k}^{ij}(m, n) V_{m+n}^{i+j-2k} + q^{2i} \tilde{c}_i(m) \delta^{ij} \delta_{m+n} , \quad (5.21)$$

with the rescaling parameter  $q$ . For the coefficients  $\tilde{g}_{2k}^{ij}(m, n)$  and further analysis of the  $q$  dependence we refer to [33]. What we are interested in is the mode dependence of the conformal charge  $\tilde{c}_i(m)$

$$\tilde{c}_i(m) = m(m^2 - 1)(m^2 - 4) \cdots (m^2 - (i + 1)^2) \tilde{c}_i \quad (5.22)$$

with the coefficient

$$\tilde{c}_i = \frac{2^{2i-2} ((i + 1)!)^2}{(2i + 1)!!(2i + 3)!!} c, \quad (5.23)$$

which shows that the conformal charge of the W algebra is already determined by the conformal charge of Virasoro algebra and thus by the topology of the underlying space. To be more precise, the charge  $c$  in (5.23) is the second Chern class of the compact manifold  $X$ . For the interesting case of  $X = K3$  this is simply  $c = 24$ , the conformal charge of the bosonic string. The mode expansion of the most general  $W_{1+\infty}(\lambda)$  algebra is far more complicated, so that we shall only consider the differences to (5.21).

For generic  $\lambda$  this W algebra has a  $\mathcal{N} = 2$  extension and not the previously considered  $\mathcal{N} = 1$  supersymmetry. But there is no contradiction, because the algebra degenerates for  $\lambda = 1/4$  to  $\mathcal{N} = 1$ . The understanding of this degeneracy offers a further connection between the improvement charge  $\alpha$ , the parameter  $\lambda$  and the supercharge of the enveloping algebra  $\text{osp}(1, 2)$  of  $W_{1+\infty}(\lambda)$ . The commutator relation of the supercharges [32]

$$[G_\alpha, G_\beta] = \left( H + \frac{1}{2} \right) \epsilon_{\alpha\beta}, \quad (5.24)$$

introduces the bosonic operator  $H$ , which commutes with any element of the algebra. As noted in Section 3, the  $AdS_3$  space has an affine  $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$  symmetry [5,6] with second Casimir operator

$$\begin{aligned} C_2 &= \frac{1}{2} \{L_1, L_{-1}\} - L_0^2 - \frac{1}{4} [G_{\frac{1}{2}}, G_{-\frac{1}{2}}] \\ &= \frac{1}{16} - \frac{1}{4} H^2 \\ &= \lambda(\lambda + 1/2). \end{aligned} \quad (5.25)$$

Introducing the Klein operator  $K$ , satisfying  $K^2 = \mathbf{1}$ , this equation can be solved for  $H$

$$H = 2 \left( \lambda - \frac{1}{4} \right) K. \quad (5.26)$$

The degeneration of the  $\mathcal{N} = 2$  superalgebra at  $\lambda = 1/4$  to  $\mathcal{N} = 1$  is obvious from (5.24) and furthermore the value  $\lambda = 0$  is a point of even higher degenerateness, as the complete fermionic part of the algebra vanishes.

The appearance of the Klein operator  $K$  has an important impact on the understanding of the W algebra from the mathematical point of view. Remember the D-brane charge dependence of the Liouville charge  $\alpha_0$  in terms of  $\beta = Q_5/Q_1$ . To compare our results with the special case of moduli spaces with only one D1-brane inserted, let us assume  $\beta \gg 1$ , which reduces (5.25) to the simple expression  $H^2 \sim Q_5/Q_1 \mathbf{1}$ , the volume of the surface of  $X$ . This relates  $K$  to its canonical class normalised by  $K^2 = 1$ . For  $X = K3$  there is no further condition on  $\beta$ , but the W algebra description of  $X = T^4$  seems only to be valid for  $\lambda = 1/4$  and fixes the relation between the number of D5-brane charges and D1-branes.

Up to now we have shown that there are W algebras that reduce to the expected conformal field theories in the limit  $Q_1 \rightarrow 1$ . But as has been already noted in [6] the commutation relations of the vertex operators in the large string limit (2.13)

$$V_{jm\bar{m}} = \gamma^{j+m} \bar{\gamma}^{j+\bar{m}} \exp\left(\frac{2j}{\alpha_+} \phi\right) \quad (5.27)$$

is basically determined by the power of the field  $\gamma(z)$  and its derivatives. Setting the conformal charge to zero, i.e.  $Q_5 = 0$ , reduces this system to the affine  $S^1$  algebra  $w_{1+\infty}$ . In this limit the difference between the insertion of  $Q_1$  vertex operators and the integration over the  $Q_1$  covering of the complex plane  $z \in \mathbf{C}$  vanishes. This strongly suggests that the structure of the W algebra remains valid also for  $Q_5 \neq 0$ . In this case the leading commutator contributions of the  $w_{\infty+1}$  algebra for the expansion modes  $v_n^i$  take the form [33]

$$[v_m^i, v_n^j] = ((j+1)m - (i+1)n) v_{n+m}^{i+j} + \frac{c_0}{12} m(m^2 - 1) \delta^{i,0} \delta^{j,0} \delta_{n+m} . \quad (5.28)$$

The complete description of the algebra  $W_{1+\infty}(\lambda)$  can be found in [32], but the structure of the leading terms is sufficient to calculate the general form of vertex operators as the characteristic function of a partition [15] along the line of (5.5). As a final step we are able to show that the whole W algebra is generated by the operator (5.1). The Virasoro mode  $a_n$  is represented by  $v_n^{-1}$  in the algebra (5.28). Thus the first derivative  $\mathcal{D}a_n = a'_n$  can be read off from (5.12) as the commutator  $[v_0^1, v_n^{-1}] = -2nv_n^0$ , which allows us to identify the scalar operator with the mode  $\mathcal{D} = -\frac{1}{2}v_0^1$  and the general form of the k-th derivative with

$$v_n^k = n^{-k} \mathcal{D}^k \cdot a_n . \quad (5.29)$$

## 5.2. The Boundary of the Hilbert Scheme

In the Introduction we explained our understanding of D-brane dynamics as a change of relative positions of branes in a  $D5/D1$  stack. The description of such a system depends on additional moduli parameters (2.6) and (2.7), which requires a more systematic analysis of the null states and its derivatives. If the moduli space of a  $D5/D1$  brane system would only depend on the product of the brane charges  $N = Q_5 Q_1$ , one simply had to sum over all possible partitions, restricted by the number of marginal deformations in the twisted sector. An explicit construction of the twisted vertex operators is not required, but only the knowledge of their conformal weights, reducing this calculation to a combinatorial problem in the  $AdS/CFT$  framework. Thus we arrive at two problems. One is the introduction of moduli parameters, which relate the cohomology classes of  $X$  to the topology of the Hilbert scheme  $\mathcal{M}$ . The second and more complicated problem is to find the generating function that reproduces the allowed partitions of D-branes in the large  $N$  limit of the  $AdS/CFT$  correspondence.

Let us concentrate on the first problem. Its solution already entered the mathematical discussion in [13,15]. Here we will motivate their construction from a more physical point of view, which already has been used in the construction of the vertex function (5.5) and the introduction of the twisted states as vectors in the space of cohomology classes. The commutator relation for the bosonic modes (3.25) can be written in the symbolic form

$$[\alpha_n^A(x), \alpha_m^B(y)] = n\delta^{AB}\delta_{n+m} \langle x, y \rangle \quad \text{for} \quad x, y \in K(X, \mathbf{Z}) \quad (5.30)$$

with the scalar product (4.16). This way the cohomology group is related to the K-group  $K(X, \mathbf{Z})$  and the RR fields [1] pulled back from the Hilbert scheme  $X^{[N]}$ . For a mathematical discussion we refer to [15]. From [1] and the discussion of Section 4.2 we know that the intersection form (4.16) generates the Néron-Severi lattice  $NS(X)$ . Now, the advantage of the string theoretic description of (5.30) becomes apparent in the different interpretations as RR fields, cohomology and K-theory. For a stack of  $D5/D1$  branes, for example, the result of the scalar product, integrated over the space  $X$  is simply  $Q_5 Q_1$ , but the string modes  $\alpha_n, \psi_m$  in (3.25) have been defined after a rescaling by  $1/\sqrt{Q_5 Q_1}$ . In the following we will therefore understand the product form  $\langle x, y \rangle$  as normalised by this factor  $N$ , which is the reason why the moduli parameter  $t$  (2.6) does not enter the vertex function (5.5). But the choice  $t = 1$  fixes all other parameters in the discussion. One example, already introduced in Section 2 is the moduli of D3-branes,  $v$ , as in (2.7). This

way the Virasoro algebra becomes an algebra over the commutative ring of the homology group of Hilbert schemes  $X^{[n]}$  with  $n = N + 1/2Q_3^2$ .

This lies at the core of the construction. Without the moduli parameters the description of the Hilbert scheme by the Virasoro algebra would only be an, although interesting, way to understand intersecting branes in two dimensions. But now the information about the surrounding space is encoded in the moduli parameters and the otherwise extremely complicated calculation of the underlying physical information in terms of algebraic geometry reduces to the simpler Virasoro algebra. In turn, the basic purpose of this algebra is to determine how to multiply the moduli parameters. Still, it takes the large  $N$  limit or, to be more precise, the infinite sum over all possible partitions to get the right picture. This again comes without surprise, as string theory is determined not by an action like field theory, but by the interactions, determined by vertex operators of the form (5.5). This hypothesis sheds a new light on many effects, typical for string theory. As an example, the Virasoro modes of a D-brane are  $a_n^i(v)$  as considered above. If our ideas are correct, the corresponding modes of an anti D-brane are  $\tilde{a}_{-n}^i(\bar{v})$  with  $\bar{v}$  the complex conjugate moduli parameter. The minus sign  $-n$  takes care of the opposite orientation of the anti D-brane compared to the original D-brane. Classically, the parameter is one of the two complex dimensions  $u = x_6 + ix_7$  or  $v = x_8 + ix_9$  with vanishing commutator. But this changes if a selfdual NS B-field in the  $(x_8, x_9)$  direction is turned on. The Néron-Severi lattice  $NS(X)$  depends only on the RR charges and does not change, but the coordinates  $(x_8, x_9)$  become noncommutative  $[x_8, x_9] = i1/2\theta$ . Let us abbreviate the commutator by a simple dot, then the two nontrivial relations are  $v \cdot v = 0$  and  $v \cdot \bar{v} = \theta$ , with all commutators with  $u$  and  $\bar{u}$  vanishing. The left and right modes of the Virasoro algebra do not commute anymore, but take the relations

$$[a_n^i(x), \tilde{a}_m^j(\bar{x})] = \theta \delta^{ij} \delta_{n-m}, \quad [L_n(x), \tilde{a}_m^i(\bar{x})] = a_{-m+n}^i(\theta) \quad \text{for } x \in K(X), \quad (5.31)$$

with an analog algebra for the open string [34]. The algebra differs in two important points. First the Kronecker delta  $\delta_{n-m}$  guarantees the condition  $N = \tilde{N}$  for the brane / anti-brane partitions, necessary to fulfil the anomaly condition. The second point is the effect of  $L_n$  on  $\tilde{a}_m^i$ . It decreases the number of antibranes by  $n$ . The missing factor of  $-m$  guarantees that the number of residual branes is identical to the number of modes. But the algebra has another interesting interpretation. The moduli parameters  $(u, v)$  correspond to two-dimensional cycles in the space  $X$ . A D5-brane and an anti D5-brane wrapping around the



same cycle  $\Sigma_1$ , denoted by  $u$ , annihilate into a D3-brane of the residual directions. Suppose, that a second cycle  $\Sigma_2$  exists with  $\Sigma_1 \cap \Sigma_2 = 1$  and parametrised by  $v$ . It would then be quite natural to identify the noncommuting parameter  $\theta$  as an additional contribution to the second moduli parameter  $v$ . This shows how the noncommutative geometry of the NS B-field and the lattice structure of K-theory are related from the moduli space point of view.

Now that the incorporation of moduli parameters has been explained, we can turn to the second problem of this section, the infinite sum over all possible D-brane partitions. In general this is as simple as the deformations of  $X$  by moduli parameters, but in practice this is the complicated part, and we have to refer to the article of M. Lehn [15] for a nontrivial example. In Section 4 we interpreted the DBI action of a brane as the total Chern class of the endomorphism bundle of the Hilbert scheme, or to be more precise, as the Chern class of the tautological bundle [15]. We already gave the physical interpretation above, namely as the interaction of a brane with a  $D5/D1$  stack. In terms of the Virasoro modes  $a_n(x)$  the propagator  $c^{-1}(u)$  acts on a single brane state by [15]

$$\begin{aligned} \mathcal{C}(u) &= c(u) \cdot a_{-1}(x) \cdot c^{-1}(u) \\ &= \sum_{n,k \geq 0} (-1)^n \binom{\text{rk}(u) - k}{n} a_{-1}^{(n)}(c_k(u)x), \end{aligned} \quad (5.32)$$

with  $u \in K(X)$  and  $x \in H^*(X, \mathbf{Z})$ . The binomial coefficient is simply obtained by a little combinatoric and the restriction that the  $n$ -fold product of a  $d$  dimensional manifold only allows a Chern class of order  $nd$  and partitions thereof. The rank of the K-group element  $u$  enters the expansion, which has a simple interpretation in string theory as the rank of the difference gauge bundle  $(E, F)$  of the branes and antibranes  $\text{rk}(u) = \text{rk}(E) - \text{rk}(F)$ . For completeness we introduce the Chern character of the tautological bundle of the Hilbert scheme, since it has an intuitive interpretation and completes the argument that justified the representation of the higher twist modes as derivatives of the primary states of the Higgs branch (5.11). The derivative (5.1) is of order one in the Chern classes and from the point of field theory its action can be understood as the insertion operator of the first Chern class  $c_1(\mathcal{F})$ . The formal Chern character  $\exp(\mathcal{F})$  defines the expansion coefficients for the generating operator  $\mathcal{D}$

$$\begin{aligned} e^{\mathcal{D}} a_{-1}(\text{ch}(u)x) &= \text{ch}(u) \cdot a_{-1}(x) \\ &= \sum_{n,k \geq 0} \frac{(-1)^n}{n!} a_{-1}^{(n)}(\text{ch}_{k-n}(u)x). \end{aligned} \quad (5.33)$$

The advantage of the Chern characters is the homogeneity of the operator expansion in comparison to the Chern classes, who interchange the degrees of the derivative  $\mathcal{D}$  and the modes  $a_n$ . For the special case of the affine  $S^1$  algebra, introduced in the previous subsection, the action of the Chern character onto a special partition can be calculated by combinatorics [15]. Take the two partitions  $\lambda = (\lambda_1, 0, \dots, 0, \lambda_n, \lambda_{n+1}, \dots)$  and  $\lambda' = (\lambda_1 + n, 0, \dots, \lambda_n - 1, \lambda_{n+1}, \dots)$  both of the same degree and note by  $a_\lambda$  the corresponding symmetric function. As shown in [15] the Chern character  $\text{ch}_{n-1}$  relates  $\lambda'$  with partitions of higher length

$$\text{ch}_{n-1} a_{\lambda'} = \binom{\lambda_1 + n}{n} a_\lambda + \dots, \quad (5.34)$$

where we again redefined all modes  $a_n$  by a minus sign. This finally justifies the identification of the derivative as the twist operator (5.11). In [15] M. Lehn comes up with another interpretation of  $\mathcal{D}$ , more appropriate from a mathematical point of view. Let  $X^{[1,1,\dots]}$  be a Hilbert scheme of order  $N$ , then the action of  $\mathcal{D}$  maps it to  $X^{[2,1,\dots]}$  the “boundary” of the Hilbert scheme.

Finally we are able to define the vertex operator in full generality with consideration of additional moduli parameters. As long as one is operating with BPS states, the effective field along example (5.5) is a holomorphic function of the complex coordinate  $z \in \mathbb{C}$  only, defined by the interaction along two dimensions.

$$\begin{aligned} e^{\Phi(z)} &= \exp \left( \int_X \mathcal{C}(u) z \right) \\ &= \sum_{n=0}^{\infty} \int_{X^{[n]}} \left( \frac{1}{n!} \mathcal{C}^n(u) \right) z^n, \end{aligned} \quad (5.35)$$

with the definition of  $\mathcal{C}(u)$  from (5.32). Here we have to be more specific with regard to the integration over the Hilbert scheme  $X^{[n]}$ . As mentioned above, the interpretation of the Hilbert scheme as the symmetric product of  $X$  is defined by a sum over all partitions. The integration over  $X^{[n]}$  now compares a specific partition with the one of  $\mathcal{C}^n(u)$ . It is quite natural to compare the Hilbert scheme, deformed by the blow ups along the singular points, with the trivial symmetric product of  $n$  copies of  $X$ . In the representation of the bosonic Virasoro algebra this is simply the mode  $\mathcal{C}(1_X) = a_1$ , with no further moduli inserted. The effective vertex operator of this special partition takes the simple form

$$e^{\Phi_1(z)} = e^{a_1 z} \cdot 1_X. \quad (5.36)$$

This is one example, where the naive correspondence between vertex operators of string theory differs from the one of the Hilbert scheme. But there is one more important case. To define the vacuum structure of the underlying D-brane configuration, one still has to determine the null mode of the field  $a_0$ , which is defined by the relation  $a_0|\chi_{0,0}\rangle = \alpha_0|\chi_{0,0}\rangle$ . The vacuum state connects the effective field picture with the topological information of  $X$  encoded in the conformal charge of the algebra. The third operator which is important for our discussion and characterises the Hilbert scheme  $X^{[0]}$  is therefore

$$e^{\Phi_0(z)} = e^{a_0 z} . \quad (5.37)$$

For all these vertex operators the incorporation of antibranes is obvious in the commutative case, as the two sectors are independent. The only crucial condition one has to consider is the restriction from K-theory for D-brane / anti D-brane interactions [34].

Up to now, we only considered the effect of interacting D-branes along two dimensions and the consequences for the residual compact space. But what about the noncompact directions as occurring in the algebra (2.12)? The Néron-Severi lattice  $\text{NS}(X)$  defines the signature of the Hilbert scheme, whereas one must not forget the original metric of the ten-dimensional spacetime. How do the two metrics combine? For example, consider the vertex operator

$$V_{-\frac{1}{2}, -\frac{1}{2}} = e^{-\frac{1}{2}\phi(z)} e^{-\frac{1}{2}\widetilde{\phi}(\bar{z})} k \cdot \Gamma_{\alpha\beta} S^\alpha S^\beta e^{ikX(z, \bar{z})} , \quad (5.38)$$

with  $\alpha$  the spinor index for the non compact directions and  $\beta$  the index from the  $\text{NS}(X)$  lattice. The problem is best studied for the Clifford algebra, which only depends on the signature and the dimension of a manifold. For simplicity we choose  $X = K3$  with  $\text{rk}(\text{NS}(K3)) = 24$ . Modular invariance of the partition function forces a space of 10 or 26 dimensions, which leaves only one possibility. Two of the 10 dimensions are fixed by the directions of the D-brane interactions, so that the residual 16 dimensions have to be “compactified”, analogous to the heterotic string. But this formal compactification has the consequence of changing the Hilbert scheme, without necessarily affecting the topology of the original manifold  $X$ . The only free field, not determined by the cohomology of  $X$ , is the NS B-field. From the point of the effective vertex operator (5.35), this additional twist can be interpreted as a torsion element in the  $\text{NS}(X)$  group; but we have no example for such a twist. The second condition to take care of, is the signature of the metric. Before compactifying 16 of the 24 dimensions of the  $K3$  lattice, one has to determine the number of timelike directions. It is not necessary to have exactly one time direction and string

theories with different signatures have been analysed [35], but the energy of such a theory may be indefinite and thus the moduli space of this manifold need not to be stable. We will simply assume, that the signature of the NS lattice is of the form  $(1, D - 1)$  to get a stable moduli space and a closed vertex operator algebra.

Having explained the most basic ideas of the construction, we will show how to apply these methods to string theory. The simplest possible generalisation of the vertex function of the  $D5/D1$  system (5.5) is the embedding of D3-branes. For the manifold  $X$  this is equivalent to the embedding of a complex curve  $\Sigma$  and thus the introduction of a holomorphic line bundle  $-\mathcal{O}(H)$  of first Chern class  $c_1 = H$ . With the introduction of the moduli parameter (2.7) a calculation similar to (5.5) becomes impracticable. We will not try to gain information from the effective vertex operator or the resulting effective Virasoro algebra, but determine the 3-point correlator instead

$$\begin{aligned}
\langle e^{\Phi_0(z=0)} \cdot e^{\Phi_1(z=1)} \cdot e^{\Phi(z)} \rangle &= \langle a_0 \cdot \left( \sum_{m=0}^{\infty} \frac{1}{m!} a_1^m \right) \cdot \left( \sum_{n=0}^{\infty} \int_{X^{[n]}} \left( \frac{1}{n!} \mathcal{C}^n(u) \right) z^n \right) \rangle \\
&= c_0 \sum_{n=0}^{\infty} \frac{1}{n!} \langle \frac{1}{n!} a_1^n | \mathcal{C}^n(u) \rangle z^n \\
&= c_0 \sum_{n=0}^{\infty} N_n z^n .
\end{aligned} \tag{5.39}$$

There are two possibilities for calculating the tree graph. The first one is a generalisation of the operator product expansion to the W algebra. But unfortunately not all coefficients are reproduced correctly, although the basic structure of the exact solution is obtained. The alternative is based on the step-by-step calculation of the coefficients  $N_n$ . After integrating over the trivial Hilbert scheme, the connection between the moduli parameter  $v$  and the Chern classes has still to be determined. For the simple case considered here, the embedding of the line bundle  $-\mathcal{O}(H)$  into  $X$  identifies the parameter as the Chern class

$$v = c(-\mathcal{O}_X(H)) = \frac{c(\mathcal{O}_X)}{c(-\mathcal{O}_H)} = 1 - H + H^2 . \tag{5.40}$$

In addition to the rank  $\text{rk}(-\mathcal{O}(H)) = -1$  of the bundle, this has to be inserted into the formula for the Chern classes of the Hilbert scheme

$$\mathcal{C}(-\mathcal{O}(H)) = \sum_{n=0}^{\infty} a_{-1}^{(n)} v^{n+1} . \tag{5.41}$$

The exact calculation with the mathematical interpretation of the individual coefficients  $N_n$  has been done in [15]. Here we will only quote the result. After the coordinate transformation

$$z = \frac{k(1-k)(1-2k)^4}{(1-6k+6k^2)^3}, \quad (5.42)$$

the generating function of the 3-point function (5.39) becomes

$$\sum_{n=0}^{\infty} N_n z^n = \frac{(1-k)^a (1-2k)^b}{(1-6k+6k^2)^c}, \quad (5.43)$$

with the abbreviations  $a = HK - 2K^2$ ,  $b = (H - K)^2 + 3\chi(\mathcal{O}_X)$  and  $c = \frac{1}{2}H(H - K) + \chi(\mathcal{O}_X)$  with the canonical bundle  $K$  and the Euler characteristic  $\chi$ . The last of the three coefficients is the Euler characteristic  $\chi(\mathcal{O}_H)$  as determined by the adjunction formula [36].

In the framework of W algebra the calculation of the coefficients  $N_n$  is not appropriate for the OPE. Instead one should consider the generating function [15]

$$\sum_{n \leq 0} N_n z^n = \exp \left( - \sum_{m > 0} \frac{(-1)^m}{m} d_m z^m \right), \quad (5.44)$$

with the coefficients  $d_m$  as calculated in [15]. For  $m > 1$  these are linear combinations of the topological values  $H^2$ ,  $HK$ ,  $K^2$  and  $\chi(X)$ . And although we have been able to reproduce the structure of the coefficients, we failed to get the correct factors by the W algebra. Thus what is still missing, is the correct incorporation of the ghost action to get a manifest BRST invariant formulation. A guiding hint shows up from the additional condition on the canonical class  $K$ , as for dimensional reasons it has to obey  $K^3 = 0$ . But we believe that even then the sum (5.44) has to be regularised, as the spin dependent conformal charge  $C_{00}(s) = -(6s^2 - 6s + 1)$  is divergent. This missing calculation scheme prevents us from a better understanding of the D-brane interactions and the difference between  $Q_1 = 1$  and  $Q_1 > 1$ . But nonetheless, the generalisation of the *AdS/CFT* to the W algebra is an important step in the understanding of M theory for finite brane configurations  $N$ .

### 5.3. Generalisation to Six Dimensions

During the entire discussion of the generalised *AdS/CFT* correspondence we assumed the supersymmetric background to be a four-dimensional Calabi-Yau manifold and the D-branes interacting along two dimensions. But this is of course neither the only possible

background nor the only possible way of D-brane interactions, and at the end of this article we will give an outlook to higher dimensional compactifications. Incidentally, this generalisation is necessary for the type I string, since the fusion of D-branes / anti D-branes takes place along four dimensions [34] instead of the considered two. In principle, the construction carries over to any complex manifold with hyper-Kähler structure and any number of moduli parameters. The four-dimensional spaces considered above allow one parameter only, as the submanifold itself has to be even dimensional. The construction of the previous subsection is therefore of quite general value. As an example, take a six-dimensional Calabi-Yau  $X$ . Again we start with a stack of black D1-branes and their dual D5-branes, wrapped around  $K3$ . For the  $S^3$  of the previous  $AdS_3 \times S^3 \times K3$  background we choose the fibration  $S^1 \times S^2$  by wrapping D3-branes around  $S^2$ . A further twist finally maps  $S^2$  to  $T^2$ . The D-brane configuration is still invariant under T-duality along the four dimensions of  $K3$  and the twisted  $T^2$ . Wrapping six directions of a D7-brane around  $T^2 \times K3$  determines a further twist of the product manifold to get the Calabi-Yau threefold as an elliptic fibration of  $K3$ . After this second twist the original charge duality of the  $D5/D1$  system, which was important in our analysis of the  $AdS/CFT$  correspondence, gets lost. Which shows that it is not possible to extend the previous construction to higher dimensions with only one Liouville field. Instead we take two copies of the  $AdS_3 \times S^1 \times T^2 \times K3$  background and introduce two Liouville fields  $(\phi_1, \phi_2)$  with the moduli  $(t_1, t_2)$  parametrising the volume of  $T^2$  and  $K3$ , respectively. With the volume of the Calabi-Yau set be one, the twist between the two manifolds is determined by the linear combination  $1 = qt_1 + pt_2$  for  $p, q \in \mathbf{Z}$ . Interactions of the two fields have a classical formulation by a  $W_3$  algebra with conformal charge  $c = qc_1 + pc_2$ . Further information about the elliptic fibration is encoded in the anomaly term, but we will not enter this discussion here.

## 6. Conclusions

In this article we dealt with the construction of an infinite Hilbert scheme on a compact manifold and in this way generalised the  $AdS/CFT$  correspondence as introduced by Maldacena [2]. Although the question of the missing states found a satisfactory solution here, many aspects have to be left to future investigations. In the Introduction we explained that the only degrees of freedom are the motions of branes in the supergravity background. Because we restricted the discussion to the  $X$  part of  $AdS_3 \times S^3 \times X$ , the effects of the

flat directions to the Hilbert scheme did not contribute and thus a better understanding of the dynamics of branes in this framework is still missing. But before these additional dimensions can be incorporated into the W algebras, better calculation technics have to be developed.

One further point, we think is important for a better understanding, is the puzzle of the D1-brane charge  $Q_1$ . Maybe, there is a deeper connection between the moduli space of Enriques surfaces and the supersymmetric string, as their torsion dependence allows more general “effective” fields as the moduli spaces of Calabi-Yau’s. An aspect connected to this is the generalisation of the DBI action. In Section 5 we introduced an iterative construction, as the number  $N$  of D-branes had to vary. But for finite  $N$  it should be possible to find a formulation in the gauge fields. For the  $D5/D1$  brane system considered above, the integral over the surface of the Hilbert scheme is classically known to be  $\beta = Q_5/Q_1$ , or from the point of K-theory, this is the difference between the bundle of D5-branes and D1-branes. But we have not been able to identify the additional contributions in string theory to justify our assumption.

Another problem concerns the space  $\mathcal{M}$ . In principle, it contains all parameters of the moduli space of M theory and so should be related to all other little string theories. This does not seem to be the case in general. As motivated by mirror symmetry, our construction is general enough to relate type IIA to IIB string theory, but we failed in the cases of the heterotic and open strings. Because the moduli space  $\mathcal{M}$  depends basically on the topology of  $X$ , only terms which do not generate a change of the topology may be added to (1.1). One natural generalisation is therefore the incorporation of an affine space  $\mathcal{A}$ . It does not contribute to the topology of  $X$ , but provides enough space to take care of additional twists in the moduli space. As generalisation of the construction (1.1) for the heterotic and open string theories we suggest

$$\widetilde{\mathcal{M}} = \sum_{N=1}^{\infty} \coprod_{\nu: \text{partition of } N} [X]^{\nu} \times [\mathcal{A}]^{N-|\nu|}. \quad (6.1)$$

What is the interpretation of  $\mathcal{A}$  in physical terms? From F theory [37] we know that IIB compactified on an elliptically fibered  $K3$  is dual to the heterotic theory on  $T^2$  with the moduli of  $K3$  encoded in the prepotential. But duality via M theory suggests first a resolution of the singularities on  $K3$  before the orbifold construction can be performed. Only then can the resolved space be blown down to a simple  $T^2$ , if possible at all. Now

suppose that the  $K3$  depends on  $N$  moduli. The most general prepotential is a linear combination of all possible resolutions of  $K3$  after the orbifolding, i.e. a formal polynomial

$$a_0 X^{[0]} + a_1 X^{[1]} + \dots a_N X^{[N]} \quad \text{with} \quad \# a_k \leq \binom{N}{k}, \quad (6.2)$$

of degree  $N$  and at most  $2^N$  coefficients, corresponding to all possible combinations of the  $N$  linear independent sections in  $K3$ . The space  $\mathcal{A}$  is then generated by the functions  $a_0, \dots a_N$ , depending on the residual moduli parameters, not contained in the corresponding symmetric sum  $X^{[k]}$ . But most elements in  $\mathcal{A}$  are zero because of the modular invariance of the elliptic fibres, so that in the case of an irreducible  $K3$  the polynomial reduces to  $j_N X^{[0]} + X^{[N]}$  with the  $j$ -function of the elliptic fibres of  $K3$ . Of course, the other direction of the duality has to work the same way. Consider the heterotic string on a singular  $T^2$ . The dual F theoretic description has to be on an elliptically fibred  $K3$  with moduli parameters generated during the blowing up of the singularities. The prepotential of IIB thus depends on a polynomial

$$b_0 \tilde{X}^{[0]} + b_1 \tilde{X}^{[1]} + \dots b_N \tilde{X}^{[N]}, \quad (6.3)$$

with functions in the dual space  $\mathcal{B}$ . What is interesting at the coefficients  $a_k$  and  $b_k$  is their similarity to cohomology elements. Furthermore, it would be interesting to study the D-brane spectrum of the Hilbert schemes  $X^{[k]}$  along [38] to understand the brane interactions analogously to the  $D5/D1$  systems studied in this article.

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